17-th Yugoslav Federal Mathematical Competition 1976

High School

Kragujevac, 1976

1-st Grade

1. Given N objects, assume that N_a of them have a property a, N_b a property b, N_c a property c, $N_{a,b}$ properties a and b, $N_{a,c,}$ properties a and c, and $N_{b,c}$ properties b and c. Prove that

$$3N + N_{a,b,} + N_{a,c} + N_{b,c} \geq 2N_a + 2N_b + 2N_c.$$

2. Prove that it is impossible to place 400 points inside a circle of radius 9 in such a way that the distance between any two of the points is greater than 1.

3. Assume that a, b, c are three real numbers such that $abc = 1$ and $a+b+c > \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Prove that exactly one of these three numbers is greater than 1.

4. A river has a shape of a straight line. A town A is located at the bank of the river, while another town B is located away from the river. Determine the place to build a port C on the river to make the transport from A to B the cheapest possible if we know that the price of transport along the river is twice smaller than the price of transport over the roads.

2-nd Grade

1. Evaluate the sum:

$$\frac{1}{2\sqrt{1}+1\sqrt{2}} + \frac{1}{3\sqrt{2}+2\sqrt{3}} + \cdots + \frac{1}{100\sqrt{99}+99\sqrt{100}}.$$

2. Given a regular hexagon of side length a, using just a straight edge construct a segment of length a/n, for $n = 2, 3, 4, \ldots$

3. Given a triangle ABC with side lengths $a = BC$, $b = CA$, $c = AB$, determine the point P inside the triangle such that the value of the expression $ax^2 + by^2 + cz^2$ is minimal, where x, y, and z denote the distances from P to the lines BC, CA, and AB.

4. Determine the largest number divisible by 11 whose all digits (in decimal expansion) are different.

3-rd Grade

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1. If $a > 1$, $b > 1$, $c > 1$ or $0 < a < 1$, $0 < b < 1$, $0 < c < 1$, prove that

$$\frac{\log_b a^2}{a+b} + \frac{\log_c b^2}{b+c} + \frac{\log_a c^2}{c+a} \geq \frac{9}{a+b+c}.$$

2. If α, β, and γ are angles of a triangle prove that

$$\sin\alpha + \sin\beta + \sin\gamma > \cos\alpha + \cos\beta + \cos\gamma.$$

3. Determine the maximal value for the ratio of the volume of a ball and the volume of a cone circumscribed around the ball.

4. Given a set S of n points ($n > 2$) in a plane, assume that for any two A, B, $\in S$, there exists $C \in S$ such that ABC is equilateral. What are the possible values for n?

4-th Grade

1. Let S be the set of all points in a plane with integer coordinates. Prove that for each positive integer n there exists a circle with center $(\sqrt{2}, 1/3)$ that contains exactly n points of the set S.

2. Closed convex curves C_1' and C_2' have perimeters x_1 and x_2, where $x_1 + x_2 = d =$ const. Curves C_1' and C_2' are similar to C_1 and C_2. The curves C_1 and C_2 have perimeters O_1 and O_2, and areas p_1 and p_2. Find x_1 and x_2 so that the sum of the areas enclosed by C_1' and C_2' is minimal.

3. Two subsets of integers are given

$$A = \{a_1, a_2, \ldots, a_n\}, \quad B = \{b_1, b_2, \ldots, b_n\}$$

in such a way that there are $x \in A$ and $y \in B$ such that $x \equiv y \pmod{2n}$. Do there always exist non-empty sets $A' \subset A$ and $B' \subset B$ such that the sum of the elements from A' and the elements from B' is divisible by $2n$?

4. All streets in a city form a rectangular grid with m "horizontal" and n "vertical" streets. What is the minimal length of the part of the grid that has to be paved, so that it is possible to go between any two intersections using paved roads only?

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