16-th Yugoslav Federal Mathematical Competition 1975

Belgrade, 1975

1-st Grade

1. Prove that for each real $a \in [5,10]$ the following equality holds:
$$\sqrt{a+3-4\sqrt{a-1}}+\sqrt{a+8-6\sqrt{a-1}}=1.$$

2. A point S is located in the interior or on one of the sides of an equilateral triangle ABC. The lines SA_1, SB_1, SC_1 are parallel to AC, AB, BC, respectively, and the points A_1, B_1, C_1 belong to BC, CA, AB, respectively. Prove that the sum $SA_1+SB_1+SC_1$ has a constant value independent on the choice of S.

3. Two cars simultaneously start from point A and move towards the point B. The first car spends half of its travel time driving at the constant speed u, and the other half driving with the constant speed v. The second car travels over the first half of the road driving at the speed u, and the second half of the road it drives at speed v. Which car will arrive faster at the destination?

4. Five zeroes and four ones are written on a circumference of a circle in arbitrary order. After that we write 0 between each pair of equal digits, and we write one between each pair of different digits. After this operation we erase the original digits. Prove that it is impossible to obtain 9 zeroes using the described operation.

2-nd Grade

1. Prove that the equation $ax^2+bx+c=0$ doesn't have rational solutions, if a, b, c are odd integers.

2. Four lines are given in a plane such that no two of them are parallel, and no three are concurrent. If a fourth line is parallel to one of the medians of the triangle determined by the first three, prove that each of the first three lines is parallel to one of the medians of the triangle determined by the other three lines.

3. A kid has shuffled the digits of a perfect sixth power and obtained the following sequence of digits: 0, 2, 4, 4, 7, 8, 9. Determine the original number?

4. Assume that n points are given in the interior of a square. We keep connecting points with each other and with the vertices of the square using non-intersecting line segments. What is the maximal number of line segments that can be drawn this way?

3-rd Grade

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1. Let n be an integer greater than or equal to 4. Prove that the n-gon determined by the midpoints of a given convex n-gon M has an area of at least half of the area of M.

2. Let S be an arbitrary point in the interior of $\triangle ABC$ with sides a, b, and c. Prove that
$$SA\cdot\frac{A}{2}+SB\cdot\cos\frac{B}{2}+SC\cdot\cos\frac{C}{2}\geq\frac{a+b+c}{2}.$$

3. Solve the equation:
$$\begin{aligned}&\left(\sqrt{\sqrt{x^2-5x+8}+\sqrt{x^2-5x+6}}\right)^x+\\+&\left(\sqrt{\sqrt{x^2-5x+8}-\sqrt{x^2-5x+6}}\right)^x \quad = \quad 2^{\frac{x+4}{4}}.\end{aligned}$$

4. What is the largest number of rooks that can be placed on a $3n\times 3n$ chessboard so that each of them is under attack by at most one of the others.

4-th Grade

1. A parabola $y=x^2$ is given. For $|x_0|>\sqrt{2}$ there are two normals to the parabola passing through $A(x_0,x_0^2)$. The feet of the perpendiculars, B and C, are different from A. Prove that the ilne BC intersects the axis of parabola at a fixed point (independent on x_0).

2. Solve the equation $1!+2!+\cdots+x!=y^z$, where x, y, and z are positive integers and $x>1$.

3. Given real numbers a_1, a_2, $\ldots$, a_n, assume that the following relations are satisfied:
$$|a_j|<M\ (j=1,2,\ldots,n),\quad a_1+a_2+\cdots+a_n=0.$$
Prove that $a_1+2a_2+3a_3+\cdots+na_n\leq\frac{n^2}{4}M$.

4. In a certain society each two members are either friends or enemies. Each two friends don't have common friends, while each two enemies have exactly two common friends. Prove that everybody must have the same number of friends in such a society.