Moldovan Team Selection Tests 2008

First Test

1. Let p be a prime number. Find all non-negative integers z and y such that $x^3 + y^3 - 3xy = p - 1$.

2. A set $\{1, 2, \dots, 3k\}$ is said to have a *property D* if it can be partitioned into disjoint triples so that in each of the triplets there is one number that is equal to the sum of the other two.

   (a) Prove that $\{1, 2, \dots, 3324\}$ has the property D.

   (b) Prove that $\{1, 2, \dots, 3309\}$ does not have the property D.

3. Let $\Gamma(I, r)$ and $\Gamma(O, R)$ be the incircle and the circumcircle of $\triangle ABC$, respectively. Consider all triangles that are simultaneously inscribed in $\Gamma(O, R)$ and circumscribed about $\Gamma(I, r)$. Prove that the centroids of all such triangles belong to a circle.

4. A non-zero polynomials $S \in \mathbb{R}[X, Y]$ is called *homogenous of degree d* if there is a positive integer d such that $S(\lambda x, \lambda y) = \lambda^d S(x, y)$ for all $x, y, \lambda \in \mathbb{R}$. Assume that $P, Q \in \mathbb{R}[X, Y]$ are two polynomials such that Q is homogenous of degree d and $P \mid Q$. Prove that P is homogenous of degree d as well.

Second Test

1. Find all pairs (x, y) of real numbers that solve the following system:

$$\begin{cases} x^3 + 3xy^2 & = 49, \\ x^2 + 8xy + y^2 & = 8y + 17x. \end{cases}$$

2. Let $a_1, \dots, a_n$ be positive real numbers whose total sum is less than or equal to $n/2$. Find the minimal value of

$$\sqrt{a_1^2 + \frac{1}{a_2^2}} + \sqrt{a_2^2 + \frac{1}{a_3^2}} + \cdots + \sqrt{a_n^2 + \frac{1}{a_1^2}}.$$

3. Let ω be a circumcircle of $\triangle ABC$ and let D be a fixed point on BC different than B and C. Let X be a variable point on BC different than D. Denote by Y the second intersection point of AX and ω. Prove that the circumcircle of $\triangle XYD$ passes through a fixed point.

4. Find the number of even permutations of $\{1, 2, \dots, n\}$ with no fixed points.

Third Test

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1. Determine a subset $A \subset \mathbb{N}$ with 5 different elements such that the sum of the squares of its elements is equal to their product.

2. Let p be a prime number and k, n be positive integers such that p and n are relatively prime. Prove that $\binom{n \cdot p^k}{p^k}$ and p are relatively prime.

3. The bisector of the angle ACB of $\triangle ABC$ intersects the side AB at D. Consider an arbitrary circle ω passing through C and D that is tangent to neither BC nor CA. Let M be the intersection of ω with BC and N the intersection of ω with CA.

   (a) Prove that there is a circle σ such that DM and DN are tangent to σ at M and N, respectively.

   (b) Denote by P and Q the intersection of the lines BC and CA with σ, respectively. Prove that the lengths of MP and NQ do not depend on the choice of ω.

4. A non-empty set S of positive integers is called *good* if there is a coloring with 2008 colors of all positive integers so that no number from S is the sum of two different positive integers (not necessarily from S) of the same color. FInd the largest value that t can take so that the set $S = \{a+1, a+2, \ldots, a+t\}$ is good for every positive integer a.