Vietnamese IMO Team Selection Test 2009

First Day

1. Denote by A_1, B_1, and C_1 the feet of perpendiculars from the vertices A, B, C of $\triangle ABC$ to BC, CA, AB. Let A_2, B_2, C_2 be the reflections of A_1, B_1, and C_1 with respect to the midpoints of BC, CA, AB, respectivley. Assume that the circumcircle of $\triangle ABC$ intersects the circumcircles of AB_2C_2, BC_2A_2, and CA_2B_2 at A_3, B_3, C_3 respectively. Prove that A_1A_3, B_1B_3, and C_1C_3 are concurrent.

2. Given a polynomial $P(x) = rx^3 + qx^2 + px + 1$ $(r > 0)$, assume that the equation $P(x) = 0$ has exactly one real root. A sequence (a_n) is defined by $a_0 = 1$, $a_1 = -p$, $a_2 = p^2 - q$, $a_{n+3} = -pa_{n+2} - qa_{n+1} - ra_n$. Prove that (a_n) contains an infinite number of negative elements.

3. Let a and b be positive integers such that none of a, b, ab is a perfect square. Prove that at most one of $ax^2 - by^2 = 1$ and $ax^2 - by^2 = -1$ has solutions in the set of positive integers.

Second Day

4. Let a, b, c be positive numbers. Find k such that:
$$\left(k+\frac{a}{b+c}\right)\cdot\left(k+\frac{b}{c+a}\right)\cdot\left(k+\frac{c}{a+b}\right) \geq \left(k+\frac{1}{2}\right)^3.$$

5. Let AB be a diameter of a circle k. Let M be a variable point inside k. The internal bisector of $\angle AMB$ intersects k at N, and the external bisector of $\angle AMB$ intersects NA, NB at P and Q. Denote by R and S the intersections of AM and BM with the circles with diameters NQ and NP respectively. Prove that the median from N of $\triangle NRS$ passes through a fixed point independent on the choice of M.

6. There are $6n+1$ mathematicians at a conference with $2n+1$ meetings. The conference room has one round table with 4 seats and n round tables with 6 seats. It is known that for any two mathematicians, the sum of the number of times they sit next to each other and the number of times they sit opposite to each other does nt exceed one.

   (a) Is such a conference possible for $n = 1$?

   (b) Determine whether such conference is possible for $n > 1$.

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