Chinese IMO Team Selection Test 2007

Time: 4.5 hours each day.

First Day

1. Points A and B lie on a circle k with center O. Let C be a point outside the circle and let CS and CT be the tangents to the circle. Let M be the midpoint of the smaller arc AB of k. The lines MS and MT intersect AB at E and F respectively. The lines passing through E and F perpendicular to AB intersect OS and OT at X and Y respectively. A line passing through C intersects the circle k at P, Q ($P \in CQ$). Let R be the intersection of MP with AB, and let Z be the circumcenter of $\triangle PQR$. Prove that X, Y, and Z are collinear.

2. A natural number x is called *good* if it satisfies: $x = p/q > 1$ with $p, q \in \mathbb{N}$, $(p,q) = 1$, and there exist constants α, N such that for any integer $n \geq N$,
$$|\{x^n\} - \alpha| \leq \frac{1}{2(p+q)}.$$
Find all good numbers.

3. There are 63 points on a circle with diameter 20. Let S be the number of triangles whose vertices are three of the 63 points and side lengths are ≥ 9. Find the maximum of S.

Second Day

4. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that
$$f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)}.$$

5. Let $x_1, \ldots, x_n$ ($n > 1$) be real numbers satisfying $A = |\sum_{i=1}^n x_i| \neq 0$ and $B = \max_{1 \leq i < j \leq n} |x_j - x_i| \neq 0$. Prove that for any n vectors $\overrightarrow{\alpha_i}$ in the plane, there exists a permutation $(k_1, \ldots, k_n)$ of the numbers $1, 2, \ldots, n$ such that
$$\left| \sum_{i=1}^{k} x_{k_i} \overrightarrow{\alpha_i} \right| \geq \frac{AB}{2A+B} \max_{1 \leq i \leq n} |\overrightarrow{\alpha_i}|.$$

6. Let n be a positive integer and let $A \subseteq \{1, 2, \ldots, n\}$. Assume that for any two numbers $x, y \in A$ the least common multiple of x and y is not greater than n. Show that
$$|A| \leq 1.9\sqrt{n} + 5.$$

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