Bulgarian Mathematical Olympiad 2002, III Round

First Day, ? April 2002

1. Find all triples (x,y,z) of positive integers such that

$$x! + y! = 15 \cdot 2^{z!}$$

Nikolai Nikolov, Emil Kolev

2. Let E and F be points on the sides AD and CD of a parallelogram $ABCD$ such that $\angle AEB = \angle AFB = 90^\circ$, and G be the point on BF for which $EG \| AB$. If $H = AF \cap BE$ and $I = DH \cap BC$, prove that $FI \perp GH$.

Ivailo Kortezov

3. Find all positive integers n for which there exist rel numbers x, y and z such that

$$x = y - \frac{1}{y^n} \quad , \quad y = z - \frac{1}{z^n} \quad , \quad z = x - \frac{1}{x^n}$$

Sava Grozdev

Second day, ? april 2002

4. Two points A and B are given on a circle, and a point C varies on it so that triangle ABC is an acute triangle. Let E and F be the orthogonal projections of the midpoint of the segment AB on AC and BC. Prove that the perpendicular bisector of the segment EF passes through a fixed point.

Alexander Ivanov

5. Let a, b and c be positive numbers such that

$$abc \leq \frac{1}{4} \qquad \text{and} \qquad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} < 9$$

Prove that there exists a triangle with sides a, b and c.

Oleg Mushkarov, Nikolai Nikolov

6. Let $n \geq 3$ be a positive integer and let $(a_1, a_2, \ldots, a_n)$ be an arbitrary n-tuple of different real numbers with positive sum. A permutation $(b_1, b_2, \ldots, b_n)$ of these numbers is called *good* if $b_1 + b_2 + \cdots + b_k > 0$ for any $k = 1, 2, \ldots, n$. Find the least possible number of *good* permutations.

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