Yugoslav IMO Team Selection Test 1982

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1. Let $p > 2$ be a prime number. For $k = 1, 2, \ldots, p-1$ we denote by a_k the remainder when k^p is divided by p^2. Prove that

$$a_1 + a_2 + \cdots + a_{p-1} = \frac{p^3 - p^2}{2}.$$

2. Find all polynomials $P_n(x)$ of the form

$$P_n(x) = n!x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + (-1)^n n(n+1),$$

with integer coefficients, such that its roots $x_1, x_2, \ldots, x_n$ satisfy $k \leq x_k \leq k+1$ for $k = 1, 2, \ldots, n$.

3. Let be given real numbers $x_i > 1$ $(i = 1, 2, \ldots, 2n)$. Prove that the interval $[0, 2]$ contains at most $\binom{2n}{n}$ sums of the form $\alpha_1 x_1 + \cdots + \alpha_{2n} x_{2n}$, where $\alpha_i \in \{-1, 1\}$ for all i.

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