Yugoslav IMO Team Selection Test 1968

Belgrade, 1968

1. Given 6 points in a plane, assume that each two of them are connected by a segment. Let D be the length of the longest, and d the length of the shortest of these segments. Prove that $\frac{D}{d} \geq \sqrt{3}$.

2. Let $n > 3$ be a positive integer. Prove that n is prime if and only if there exists a positive integer α such that $n! = n(n-1)(\alpha n + 1)$.

3. Each side of a triangle ABC is divided into three equal parts, and the middle segment in each of the sides is painted in green. In the exterior of $\triangle ABC$ three equilateral triangles are constructed, in such a way that the three green segments are sides of these triangles. Denote by A', B', C' the vertices of these new equilateral triangles that don't belong to the edges of $\triangle ABC$, respectively. Let A'', B'', C'' be the points symmetric to A', B', C' with respect to BC, CA, AB.

   (a) Prove that $\triangle A'B'C'$ and $\triangle A''B''C''$ are equilateral.

   (b) Prove that ABC, $A'B'C'$, and $A''B''C''$ have the common centroid.

4. If a polynomial of degree n has integer values when evaluated in each of k, $k+1$, $\dots$, $k+n$, where k is an integer, prove that the polynomial has integer values when evaluated at each integer x.

5. Let n be an integer greater than 1. Let $x \in \mathbb{R}$.

   (a) Evaluate $S(x,n) = \sum (x+p)(x+q)$, where the summation is over all pairs (p,q) of different numbers from $\{1,2,\dots,n\}$.

   (b) Do there exist integers x, for which $S(x,n) = 0$.

6. Prove that the incenter S coincides with the circumcenter S' of a tetrahedron if and only if each pair of opposite edges are of equal length.

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