38-th Yugoslav Federal Mathematical Competition 1997

High School

Niš, April 12, 1997

Time allowed 4 hours.
Each problem is worth 25 points.

1-st Grade

1. A pawn moves on points with integer coordinates in Cartesian plane according to the following rules.

   (a) Initially, the pawn is placed at point (m,n).

   (b) If the pawn is at point (x,y) in some moment, then he steps to point $(x+1,y)$, $(x,y+1)$, $(x-1,y)$, $(x,y-1)$, depending on whether the remainder of $x+y$ when divided by 4 is $0,1,2,3$, respectively.

   If the pawn arrives at point $(0,1997)$ after 1997 steps, find all possible points (m,n).

2. Let O be an interior point of a triangle ABC. Lines OA,OB,OC meet the corresponding sides at points P,Q,R, respectively. Find the minimum possible value of $\frac{AO}{OP}\cdot\frac{BO}{OQ}\cdot\frac{CO}{OR}$, and determine the points O for which this value is attained.

3. In a triangle ABC, CD is the altitude, E the midpoint of side AB, and P,Q are the feet of perpendiculars from A and B to the bisector of angle ACB, respectively. Show that points D,E,P,Q lie on a circle.

4. Prove that among the numbers of the form $\left[2^{k+1/2}\right]$, where k is a natural number, there are infinitely many even numbers.

2-nd Grade

1. A line s and points A,B on the same side of s are given in the plane. For a variable point M, let A_1 and B_1 be the orthogonal projections from A and B to lines MB and MA, respectively. Find the position of M that minimizes the length A_1B_1.

2. Each edge of a convex polyhedron is denoted by a $+$ or a $-$. Prove that there must exist a vertex of the polyhedron such that among the angles at that vertex less than four have the rays denoted by different signs.

3. Let $S(n)$ denote the sum of (decimal) digits of a natural number n. Does there exist n such that $S(n)=1997$ and $S(n^2)=1997^2$?

4. There are three schools, each of which is attended by n pupils. Every pupil knows at least $n+1$ pupils from the remaining two schools. Prove that there exist three pupils, one from each school, who know each other. (Acquaintance is a symmetric relation.)

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

3-rd and 4-th Grades

1. Does there exist a natural number n such that the set $\{n, n+1, \ldots, n+1997\}$ can be partitioned into (disjoint) subsets $A_1, A_2, \ldots, A_k$, $k \geq 2$, with the same products of elements?

2. Consider the polynomial $P(x) = x^6 - 2x^5 + x^4 - 2x^3 + x^2 - 2x + 1$. Prove that exactly four roots of $P(x)$ have module 1.

3. Let be given a tetrahedron $ABCD$ of volume V. Points A_1, B_1, C_1 are taken on sides AD, BD, CD respectively so that the plane $A_1B_1C_1$ contains the centroid of the tetrahedron. Let V_1, V_2, V_3 denote the volumes of tetrahedra $AA_1B_1C_1$, $BA_1B_1C_1$ and $CA_1B_1C_1$, respectively. Find the minimum possible value of $V_1 + V_2 + V_3$.

4. Five jugs are numbered $0, 1, 2, 3, 4$. The jug 0 is empty, whereas each of the remaining jugs contains several coins. There are r_i coins in the jug i. Players A and B play the following game. They alternately move the coins, and a player in turn can move arbitrarily many coins (at least one) from any jug to the jug numbered with a number less by 1. Player A begins. The player after whose move all the coins are in the jug 0 gains all the coins. Which player has a winning strategy?