35-th Yugoslav Federal Mathematical Competition
1994

High School
Ivanjica, April 16, 1994

Time allowed 4 *hours.*
Each problem is worth 25 *points.*

1-st Grade

1. Given positive numbers a,b,c,x,y,z such that $a+x=b+y=c+z=1994$, prove that $ay+bz+cx<1994^2$.

2. Let D be the point on the side AB of the triangle ABC different from A and B. Let r,r_1,r_2 be the radii of circles inscribed into $\triangle ABC$, $\triangle ADC$ and $\triangle DBC$ respectively. Prove that $r<r_1+r_2$.

3. Let M,N,P,Q be respectively the midpoints of the sides AB,BC,CD,DA of a convex quadrilateral $ABCD$. Segments MP and NQ intersect at the point O. Prove that the sum of the areas of the quadrilaterals $AMOQ$ and $CPON$ is equal to the sum of the areas of the quadrilaterals $BNOM$ and $DQOP$.

4. On the blackboard there are written 1993 digits 0, 1994 digits 1 and 1995 digits 2. On each move it is allowed to erase two different digits and to write one of $1,2,3$ which is not deleted on that move.

   (a) Is it possible that after certain number of moves only zeroes remain on the board?

   (b) If there is only one digit on the board, determine that digit.

2-nd Grade

1. Determine all numbers $b<100$ such that 2101 written in the system with the base b is square of an integer.

2. Let
$$f(x,y)=\begin{cases} (y+1)^2-x, & \text{if } x\le y,\\ x^2+y+1, & if\text{x¿y.}\end{cases}$$
   Solve the equation $f(x,y)=1994$ in the set of nonnegative integers.

3. In a cyclic quadrilateral each vertex is connected by a line with the orthocenter of the triangle determined by the other three vertices. Prove that the obtained four lines intersect at one point.

4. (a) Given a convex hexagon whose area is S and arbitrary line l in the plane of the hexagon. Prove that there exist a triangle inscribed in the given hexagon such that one of its sides is parallel to l and the area of the traingle is greater than or equal to $\frac{3S}{8}$.

   (b) Determine the greatest possible area of the triangle inscribed in a regular hexagon of the area S, such that one side of the triangle is parallel to the side of the hexagon.

   (A triangle is inscribed in the hexagon if all of its vertices belong to the sides of a hexagon.)

3-rd and 4-th Grades

1. The lengths of the sides of a convex quadrilateral are natural numbers. If each of the numbers divides the sum of the remaining three, prove that among the numbers there are at least two equal.

2. Let q be a prime number greater than 5 and $1 \leq p < q$. Suppose that the number $\frac{p}{q}$ has purely periodical decimal expansion with the period $2n$. Prove that the sum of the number formed by the first n digits of the period and the number formed by the last n digits of the period is equal to $10^n - 1$.

3. Given a trihedral $Oabc$, prove that on the edges a, b, c there exist points A, B, C respectively (different from O) such that $\angle OAB = \angle OBC = \angle OCA$.

4. Let S be the set of all arrays $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ whose elements are from the set $\{0, 1, 2\}$. Let *elementar transform* of the array $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be the replacing of one element α_j with $\beta_j \in \{0, 1, 2\}$ such that no one of the numbers α_j, β_j appears in the array $(\alpha_1, \alpha_2, \ldots, \alpha_{j-1})$. Prove that each array from S can be obtained from $(0, 0, \ldots, 0)$ with finitely many elementar transforms.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com