34-th Yugoslav Federal Mathematical Competition 1993

High School

Novi Sad, April 17, 1993

Time allowed 4 *hours.*
Each problem is worth 25 *points.*

1-st Grade

1. Square of the size 10 is divided onto 100 unit squares. At the begining, 9 of these squares are black. After each year all squares that have two black neighbours become black (two squares are neighbours if they share an edge). Is it possible to place the initial 9 black squares in such a way that after certain number of years all squares become black?

2. Prove that the following inequality holds for positive numbers a and b:

$$\frac{(a-b)^2}{2(a+b)} \leq \sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab} \leq \frac{(a-b)^2}{4\sqrt{ab}}.$$

3. Convex pentagon $ABCDE$ satisfies $\angle ABC = \angle ADE$ and $\angle AEC = \angle ADB$. Prove that $\angle BAC = \angle DAE$.

4. Let ABC be an acute-angled triangle, and D an arbitrary point of the edge BC different from B and C. The outside common tangent to the circumcircles of $\triangle ABD$ and $\triangle ACD$ different from BC intersects the segment AD in the point E. Prove that the length of AE doesn't depend on the choice of D.

2-nd Grade

1. Given 1993 positive real numbers, product each 12 of them is not less than 1. Prove that the product of all these numbers is not less than 1.

2. Let $A_1A_2\dots A_{12}$ be a regular 12-gon. The vertex A_{12} is denoted by -1, and the other vertices are denoted by number 1. At each move it is allowed to choose an isosceles rectangular triangle whose vertices are among $A_1, A_2, \dots, A_{12}$ and change all the numbers at the vertices of the triangle. Is it possible to obtain the configuration with all numbers equal to -1 after a finite number of moves?

3. Let A_1 and C_1 be the midpoints of the sides BC and CA, respectively, of the triangle ABC and T the centroid of the triangle. If the circle can be inscribed into the quadrilateral A_1BC_1T, prove that the triangle ABC is isosceles.

4. Let M and N be the points of the sides AC and BC of the regular triangle ABC such that $MN \| AB$. If D is the orthocenter of the triangle MNC and E the midpoint of AN, determine the angles of the triangle BDE.

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3-rd and 4-th Grades

1. A function $f:\mathbb{R}\to\mathbb{R}$ is given by:

$$f(x)=\begin{cases} 0, & \text{if } x=\dfrac{\pi}{2}+k\pi,\ k\in\mathbb{Z},\\ \dfrac{1}{2+\tan^2 x}, & \text{otherwise.}\end{cases}$$

   Prove that the function $g(x)=f(x)+f(ax)$ is periodic if and only if a is a rational number.

2. Given seven nonnegative real numbers whose sum is 1. Prove that all these numbers can be placed on a circle such that the sum of all products of adjacent numbers is less than or equal to $\frac{1}{7}$.

3. Let A be the set of all 11-tuples whose elements are 0 and 1. Elements of the set A are placed in a sequence in the following way:

   1° First element of the sequence is $(0,0,\dots,0)$.

   2° $(n+1)$-st element of the sequence is obtained from the n-th element changing the rightmost possible digit which will lead to an 11-tuple that hasn't appeared already in the sequence.

   Find the 11-tuple on the 1993-rd position.

4. Equilateral triangles BCB_1, CDC_1, DAD_1 are constructed outside the convex quadrilateral $ABCD$. If P aqnd Q are, respectively, the midpoints of B_1C_1 and C_1D_1 and R the midpoint of AB, prove that the triangle PQR is equilateral.