28-th Yugoslav Federal Mathematical Competition 1987

High School

Titov Vrbas, April 1987

1-st Grade

1. Prove the following inequality for nonnegative numbers a and b:
$$\frac{(a+b)^2}{2}+\frac{a+b}{4}\geq a\sqrt{b}+b\sqrt{a}.$$
2. Let ABC be a triangle with an obtuse angle at A, and let $a=BC$, $b=CA$ and h_a,h_b be the altitudes from A,B respectively. Prove that $a+h_a>b+h_b$.
3. For a natural number n, find the number of solutions to the equation
$$x^2-[x^2]=(x-[x])^2$$
with $1\leq x\leq n$.
4. Every vertex of a cube is assigned one of the numbers $-1,0,1$, and each face is assigned the product of the numbers at its four vertices. Can the sum of the 14 assigned numbers be equal to (a) 7; (b) 0?

2-nd Grade

1. Prove that there is an infinity of prime numbers p for which the equation $x^2+x+1=py$ has an integer solution (x,y).
2. In a cyclic quadrilateral $ABCD$, M is the intersection point of the perpendiculars at A to AB and at D to CD, and N is the intersection point of the perpendiculars at B to AB and at C to CD. Prove that the intersection of diagonals AC and BD lies on line MN.
3. Prove that if a quadrilateral is tangent, then:
   (a) The incircles of the two triangles into which a diagonal divides the quadrilateral touch each other.
   (b) The tangency points of the incircles of these two triangles with the sides of the quadrilateral lie on a circle.
4. Let $P(x)$ be a polynomial of seventh degree with integer coefficients which takes values from the set $\{-1,1\}$ at seven distinct integer points. Prove that $P(x)$ is irreducible over polynomials with integer coefficients.

3-rd and 4-th Grades

1. Let $a_1, a_2, \ldots, a_n$ be positive real numbers with the product 1. Show that
$$(4+a_1)(4+a_2)\cdots(4+a_n) \geq 5^n.$$

2. Suppose a and m are natural numbers and x an integer such that m divides a^2x-a. Prove that there is an integer y such that m divides both a^2y-a and ay^2-y.

3. Let be given n points in space, every four of which form a nondegenerate tetrahedron of volume not exceeding 1. Prove that there exists a tetrahedron of volume 27 which contains all the given points (in its interior or on its boundary).

4. Let X be the set of all finite sequences of 0 and 1 and $f : X \to X$ be a mapping defined as follows: for $x \in X$, $f(x)$ is obtained when each 1 and each 0 in the sequence x are replaced with 01 and 10, respectively. How many subsequences 00 do occur in the sequence
$$\underbrace{f(f(\cdots f}_{n \text{ times}}(1)\cdots))?$$

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