25-th Yugoslav Federal Mathematical Competition 1984

High School

Smederevska Palanka, April 1984

1-st Grade

1. The number a is obtained by writing numbers $1,2,\dots,101$ one after another. Prove that a is a composite number. Is a a perfect square?

2. Suppose a,b,c are three pairwise distinct numbers which satisfy

$$\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}=0.$$

Prove that $\frac{a}{(b-c)^2}+\frac{b}{(c-a)^2}+\frac{c}{(a-b)^2}=0$.

3. Let O be a point inside triangle ABC and let the lines through O which are parallel to CA,AB,BC meet AB,BC,CA at K,L,M, respectively. Lines CK and AL, AL and BM, BM and CK intersect at points P,Q,R respectively. Prove that the sum of the areas of triangles AKP,BLQ and CMR is equal to the area of triangle PQR.

4. Each of the 25 cells of a 5×5 table is colored in one of two colors. Prove that there exist four cells of the same color whose centers form a rectangle with sides parallel to the sides of the big square. Prove that the statement is false for a 4×4 table.

2-nd Grade

1. Let p_n denote the n-th prime number and let $\pi(n)$ be the number of prime numbers not exceeding n. If

$$A=\{n+p_n \mid n\in\mathbb{N}\} \quad \text{and} \quad B=\{n+\pi(n)+1 \mid n\in\mathbb{N}\},$$

prove that $A\cap B=\emptyset$ and $A\cup B=\mathbb{N}\setminus\{1\}$.

2. If real numbers x,y,z satisfy the equalities

$$x+y+z=2 \quad \text{and} \quad xy+yz+zx=1,$$

show that they lie in the interval $[0,4/3]$.

3. In a convex quadrilateral $ABCD$ it holds that $\angle ABD=50^\circ$, $\angle ADB=80^\circ$, $\angle ACB=40^\circ$ and $\angle DBC=\angle BDC+30^\circ$. Compute $\angle DBC$.

4. Any two cities in a country are connected by a direct one-way air route. Prove that there is a city from which one can reach any other city with at most one change.

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3-rd and 4-th Grades

1. Determine a sequence (a_n) which satisfies the condition

$$1+\sum_{d|n}(-1)^{n/d}a_d = 0, \qquad n = 1,2,3,\dots$$

2. Show that for every natural number n the equation

$$\left(\frac{\sqrt{5}-1}{2}\right)^n x + \left(\frac{\sqrt{5}-1}{2}\right)^{n+1} y = 1$$

has exactly one solution in integers.

3. Let $ABCD$ be a given quadrilateral. Prove that if there is a point P in the plane such that triangles ABP and CDP are equally oriented isosceles right triangles with the right angles at P, then there is a point Q such that triangles BCQ and DAQ are equally oriented isosceles right triangles with the right angles at Q.

4. Let S be a set of n elements. Find the greatest m for which there is a family $\{S_1, S_2, \dots, S_m\}$ of distinct nonempty subsets of S such that the intersection of any three sets from the family is empty.