22-nd Yugoslav Federal Mathematical Competition 1981

High School
Ohrid, 1981

1-st Grade

1. Let a,b,c be positive integers such that $a+c$ and $b+c$ are squares of two consecutive natural numbers. Prove that $ab+c$ and $ab+a+b+c$ are also squares of two consecutive natural numbers.

2. In a triangle, the altitude from one vertex, the angle bisector from another vertex, and the median from the remaining vertex determine a new triangle. Show that this new triangle cannot be equilateral.

3. The first four terms in a sequence are $1,9,8,1$. Every consequent term is equal to the units digit of the sum of the preceeding four terms.

   (a) Does the quadruple $1,2,3,4$ occur in the sequence?

   (b) Does the initial quadruple occur ever again?

4. A mouse nibbles a piece of cheese of the shape of a cube of side 3. The cheese cube is divided into 27 unit cubes. The mouse eats the cheese according to the following rule: it starts at one of the corner unit cubes, and, after eating a whole unit cube, passes to one of the unit cubes which shares a face with the just eaten one. Can the mouse eat the whole piece of cheese with the central unit cube as the last one to be eaten?

2-nd Grade

1. Let a,b,c be integers with $a>0$. Suppose that the equation $ax^2+bx+c=0$ has two distinct solutions in the interval $(0,1)$. Show that $a \geq 5$ and give an example of such an equation with $a=5$.

2. The diagonals of a convex quadrilateral divide the quadrilateral into four triangles whose areas are integers. Prove that the product of these four integers is a perfect square.

3. Find all pairs (x,y) of integers which satisfy
$$y^4 - x(x+1)(x+2)(x+3) = 1.$$

4. The set $\{1,2,\dots,100\}$ is partitioned into seven mutually disjoint subsets. Prove that there exist four numbers a,b,c,d in one of these subsets, at least three of which are distinct, such that $a+b=c+d$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
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3-rd Grade

1. Prove that for each positive integer n, the number $\tan^{2n} 15^\circ + \cot^{2n} 15^\circ$ can be written as the sum of the squares of three consecutive positive integers.

2. Three similar triangles KPQ, QLP, PQM are constructed on the same side of line PQ so that

$$\angle QPM = \angle PQL = \alpha, \quad \angle PQM = \angle QPK = \beta, \quad \angle PQK = \angle QPL = \gamma,$$

where $\alpha + \beta + \gamma = 180^\circ$. Prove that the triangle KLM is similar to the three given triangles.

3. Let S_1 be the sequence of positive integers: $1, 2, 3, \dots$. For each $n \in \mathbb{N}$, the sequence S_{n+1} is obtained by increasing by 1 every term of sequence S_n which is divisible by n. Thus, for example, $S_2 = (2, 3, 4, 5, 6, \dots)$ and $S_3 = (3, 3, 5, 5, 7, 7, \dots)$. Prove that the first $n-1$ terms of sequence S_n are equal to n if and only if n is prime.

4. Let $A_1, A_2, \dots, A_{1966}$ be subsets of a finite set M such that $|A_j| > |M|/2$ for each j. Show that there exist elements $x_1, x_2, \dots, x_{10}$ of M such that each of the A_j's contains at least one of $x_1, x_2, \dots, x_{10}$.

4-th Grade

1. Prove that if a line divides a triangle into two parts of equal areas and perimeters, then this line passes through the incenter of the triangle.

2. Given positive real numbers a, b, find the maximum value of $\left|\dfrac{x+y}{1+x\bar{y}}\right|$, where x, y are complex numbers with $|x| = a$ and $|y| = b$.

3. Define $F_n = a^n \sin nA + b^n \sin nB + c^n \sin nC$, where a, b, c, A, B, C are real numbers and $A + B + C$ is a multiple of π. Prove that if $F_1 = F_2 = 0$, then $F_n = 0$ for each natural number n.

4. A set $S = \{1, 2, \dots, n\}$ has been partitioned twice: the first time into m nonempty subsets, and the second time into $m + k$ nonempty subsets ($k > 0$). Prove that there are at least $k + 1$ elements of S which have belonged to more numbered subsets in the first partition, than in the second partition.

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