21-st Yugoslav Federal Mathematical Competition 1980

High School
Kumrovec, 1980

1-st Grade

1. A pencil costs an integer number of cents. The total price of 9 pencils is more than 11, but less than 12 euros. The total price of 13 pencils is more than 15, but less than 16 euros. What is the price of one pencil?

2. Consider numbers $1, 12, 123, \dots, 1234567890, 12345678901, \dots$. Prove that at least one of these numbers is divisible by 1981.

3. Let D be the point on side BC of a triangle ABC such that $DC = 2BD$. Given that $\angle ABC = 45^\circ$, $\angle ADC = 60^\circ$, determine the angles of triangle ABC.

4. There are 1980 crossings in a city, and through each crossing pass three streets. There is a round bus route passing through each crossing exactly once. It has been decided that, along each street, trees of exactly one of the three sorts, chestnut, birch and linden, be planted. Prove that this can be done so that at each crossing there are trees of all the three sorts.

2-nd Grade

1. Determine all integers x for which $x^2 + 3x + 24$ is a perfect square.

2. A circle is inscribed in a rhombus $ABCD$. A tangent to the circle meets sides BC and CD at points M and N. Prove that the area of triangle AMN is constant.

3. Is it possible to cover a square K of side 7 with eight squares of side 3,

   (a) under the condition that the sides of the small squares must be parallel to sides of square K;

   (b) without this condition?

4. Let $1 \le a_1 < a_2 < \cdots < a_{19} \le 200$ and $1 \le b_1 < b_2 < \cdots < b_{21} \le 200$ be natural numbers. Prove that one can select numbers a_i, a_j, b_p, b_q among them so that $a_j - a_i = b_q - b_p > 0$.

3-rd Grade

1. Solve the equation $x^{5-x} = (6-x)^{1-x}$ in the set of positive integers.

2. Let be given 18 segments of lengths $x_1, x_2, \dots, x_{18}$ satisfying $1 \le x_1 \le x_2 \le \cdots \le x_{18} \le 1980$. Show that a triangle can be formed from some three of these segments.

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3. The diagonals of a convex quadrilateral $ABCD$ intersect at point S. Given that

$$\angle SAB = \angle SBC = 30^\circ, \qquad \angle SCD = \angle SDA = 45^\circ,$$

find the angle between the diagonals of the quadrilateral.

4. Find all polynomials $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with $a_j \in \{-1, 1\}$ for each j, whose all roots are real.

4-th Grade

1. An ellipse is given by the equations $x = a\cos t$, $y = b\sin t$, where $a \neq b$. Prove that the points determined by parameters t_1, t_2, t_3, t_4 (assuming these are pairwise distinct) lie on a circle if and only if there is an integer k such that $t_1 + t_2 + t_3 + t_4 = 2k\pi$.

2. Let S be a set of n real numbers and let T be the set of all possible sums of k distinct numbers in S, where $n \geq k$. Prove that set T has at least $k(n-k)+1$ elements.

3. Let a be a positive integer. The sequence (a_n) is defined by $a_0 = a$ and, if $a_n = c_0 + 10c_1 + \cdots + 10^k c_k$, where $c_0, \dots, c_k \in \{0, 1, \dots, 9\}$, then

$$a_{n+1} = 2c_0 + c_1 + 10c_2 + \cdots + 10^{k-1}c_k.$$

Which numbers occur in the sequence (a_n) infinitely often?

4. Function $f : [0,1] \to [0,1]$ satisfies $0, 1 \in f([0,1])$ and

$$|f(x) - f(y)| \leq \frac{|x - f(x)| + |y - f(y)|}{2} \quad \text{for all } x, y \in [0,1].$$

Prove that there is exactly one number $x \in [0,1]$ such that $f(x) = x$.

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