15-th Yugoslav Federal Mathematical Competition 1974

Belgrade, 1974

1-st Grade

1. Graphically express the set of points (x,y) such that
$$||x|-1|+||y|-1| \leq 1.$$

2. Three lines p, q, and r intersect in three different points. Using a ruler and a compass determine the line m perpendicular to r, such that the segment determined by the intersections of p and q with m, and the segment determined by the intersections of q and r with m are congruent.

3. Find a six digit positive integer whose products with 2, 3, 4, 5, 6 in some order form different six-digit numbers that can be obtained by rearranging the digits of the original number.

4. There are 1975 children on a circle. They play the following game: First child remains on a circle, the second goes away, the third stays, the fourth goes away, etc, until the last child remains on the circle. What is the number of the last child?

2-nd Grade

1. Let m and n be given positive numbers. Prove that
$$\begin{aligned} x_1 = x_3 = x_5 = \cdots = x_{99} &= m \\ x_2 = x_4 = x_6 = \cdots = x_{100} &= n \end{aligned}$$
is the only positive solution to the system of equations
$$\begin{gathered} nx_1 = x_2x_3 = x_4x_5 = \cdots = x_{98}x_{99} = mx_{100} \\ x_1+x_2 = x_3+x_4 = x_5+x_6 = \cdots = x_{99}+x_{100} = m+n. \end{gathered}$$

2. Solve the equation $x^2+xy+y^2 = x^2y^2$ in the set of positive integers.

3. Assume that $A_1A_2A_3A_4A_5A_6$ is a hexagon inscribed in a circle k. Assume further that A_1A_4, A_2A_5, A_3A_6 are the diameters of k. A point P that belongs to k is chosen to be different from the vertices of the hexagon. Let Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 be the orthogonal projections of P to A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_1, respectively.

   (a) Prove that the lines determined by the arbitrary two adjacent sides of the hexagon $Q_1Q_2Q_3Q_4Q_5Q_6$ are perpendicular.

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(b) Prove that P, the center O of k, and the midpoints R_1, R_2, R_3 of Q_1Q_4, Q_2Q_5, Q_3Q_6 belong to a circle.

3-rd Grade

1. Solve the equation:
$$\sqrt{x^2+x}+\sqrt{1+\frac{1}{x^2}}=\sqrt{x+3}.$$

2. Find the angle α, if:
$$\cot\alpha=2+\sqrt{a}+\sqrt{b}+\sqrt{c},\quad \cot 2\alpha=2+\sqrt{a},$$
where a, b, c are positive integers non divisible by 4, and $\sqrt{a}$ and $\sqrt{bc}$ are irrational numbers.

3. Given a right circular cone with vertex V, center of the base S, assume that β is the angle at V of the triangle determined by the intersection of the cone with a plane passing through the axis of the cone. Two tangent planes of the cone touch the cone along the lines VA and VB, where A and B belong to the base. If the angle between these two planes is equal to α, evaluate $\angle ASB$.

4. Find the maximum of the product of positive integers whose sum is equal to the given positive integer n.

4-th Grade

1. Find a rational number with infinite decimal expansion from the interval $\left(\frac{1}{4},\frac{1}{3}\right)$, and the sum of the digits of its period in the decimal expansion is by 12 bigger that the square of the number.

2. Find all positive integers n, such that some three consecutive coefficients in the expansion $(a+b)^n$ form an arithmetic sequence.

3. Find the tangent of the ellipse $b^2x^2+a^2y^2=a^2b^2$ whose length of the segment between the centers is minimal.

4. The first row of the 8×8 chessboard contains 8 red coins. The last row contains 8 green coins. Two players play the following game: Players R and G play the following game: Players alternate their turns; R starts the game and always moves red coins; G always moves green coins. In each of the moves each of the players chooses one of his/her coins, and moves it in its column by at least one square. However, player is not allowed to jump with its coin over the coin of the opponent. Prove that Green has the winning strategy.