40-th Yugoslav Federal Mathematical Competition 2000

High School

Negotin, April 15, 2000

Time allowed 4 *hours.*
Each problem is worth 25 *points.*

1-st Grade

1. Initially, there is one amoeba in a test-tube. Every second, one of the following two changes happens: either a few amoebas divide into seven new ones each, or exactly one dies. After how many seconds at least can there be exactly 2000 amoebas in the test-tube?

2. Let
$$S = 1 + \frac{1}{1+\frac{1}{1+2}} + \cdots + \frac{1}{1+\frac{1}{1+2}+\cdots+\frac{1}{1+2+\cdots+2000}}.$$
Prove that $S > 1003$.

3. Lines a,b,c, parallel to sides BC,CA,AB of a triangle ABC respectively, are drawn through a point O inside the triangle. Let a meet AB,AC at C_2,B_1, b meet BC,BA at A_2,C_1, and c meet CA,CB at B_2,A_1, respectively. Prove that triangles $A_1B_1C_1$ and $A_2B_2C_2$ have equal areas.

4. All vertices of a polygon in a coordinate plane are integer points, and all its sides have integer lengths. Prove that its perimeter is even.

2-nd Grade

1. Let $ABCD$ be an inscribed quadrilateral. Prove that
$$|AB-CD| + |BC-DA| \geq 2|AC-BD|.$$

2. Given $n \in \mathbb{N}$, how many sequences $(x_1,x_2,\dots,x_n)$ of $0,1,2,3$ are there, such that for any $i = 1,2,\dots,n-1$, the ordered pair (x_i,x_{i+1}) is not one of the pairs $(1,2),(1,3),(3,2),(3,3)$?

3. Among the points corresponding to numbers $1,2,\dots,2n$ on the real line, n are colored in blue and n in red. Let $a_1 < a_2 < \cdots < a_n$ be the blue points and $b_1 > b_2 > \cdots > b_n$ be the red points. Prove that the sum
$$|a_1 - b_1| + \cdots + |a_n - b_n|$$
does not depend on coloring, and compute its value.

4. Prove that every positive rational number can be written in the form $\dfrac{a^3+b^3}{c^3+d^3}$, where a,b,c,d are positive integers.

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3-rd and 4-th Grades

1. Let a and b be skew lines determined by two sides of a cube, and let $P \in a$ and $Q \in b$ be points such that PQ is the common perpendicular to a and b. Find all possible values of MP/PN, where M, N are the vertices of the cube on a.

2. Numbers $1, 2, \dots, 64$ are written in a 8×8 board. For every two numbers a, b with $a > b$ in the same row or column, the ratio a/b is calculated. The *characteristic* of the board is defined as the least of these ratios. Find the greatest possible value of a characteristic.

3. Denote by S the set of all primes p such that the decimal representation of $1/p$ has the fundamental period divisible by 3. For each $p \in S$, we can write $1/p = 0.\overline{c_1 c_2 \dots c_{3r}}$, where $3r$ is the fundamental period of p; we define
$$f(k,p) = a_k + a_{k+r} + a_{k+2r}$$
for every $k = 1, 2, \dots, r$. Determine the maximum possible value of $f(k,p)$.

4. Prove that for every positive integer n it holds that:
$$\sum_{i=0}^{n} (-1)^i \binom{2n-i}{i} 2^{2n-2i} = 2n+1.$$