Vietnamese IMO Team Selection Test 1999

First Day – May 9

1. Let p be an odd prime number such that $2^h - 1$ is not divisible by p for $h = 1, 2, \dots, p-2$. For an even integer a_0 with $p/2 < a_0 < p$, define the sequence (a_n) inductively by $a_{n+1} = p - b_n$, where b_n is the largest odd divisor of a_n. Prove that the sequence (a_n) is periodic and find its period.

2. Find all polynomials $P(x)$ of degree 1999 with real coefficients such that for some real number a,
$$P(x)^2 - 4 = a(x^2-4)P'(x)^2.$$

3. Let be given a convex polygon $\mathscr{H}$. Prove that there exist six distinct points $A_1, \dots, A_6$ on its boundary such that:

   (i) the lines A_1A_2, A_3A_6 and A_4A_5 are parallel;

   (ii) the line A_3A_6 is equidistant from A_1A_2 and A_4A_5;

   (iii) $A_1A_2 = A_4A_5 = c \cdot A_3A_6$, where $0 < c < 1$ is a given constant.

Second Day – May 10

4. Define $H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Let (u_n) be a sequence of positive real numbers and $U(n) = u_1 + u_2 + \cdots + u_n$. For $n \in \mathbb{N}$, let k_n be the least positive integer such that $H(k_n) \geq U(n)$. Prove that $\lim_{n\to\infty} \frac{k_{n+1}}{k_n}$ exists if and only if $\lim_{n\to\infty} u_n$ exists.

5. Let ABC be a given triangle. A circle k_a is internally tangent to the circumcircle of ABC and touches AB and AC at M_1 and N_1, respectively. Points M_2, N_2 and M_3, N_3 are analogously defined. Prove that the segments M_1N_1, M_2N_2 and M_3N_3 have the common midpoint.

6. Let p be an odd prime number. At each vertex of a regular p-gon there is a monkey. A person having p peanuts walks around the polygon. At first he gives the first peanut to some monkey. Thereafter, after giving the k-th peanut, he starts counting clockwise from his current position and gives the next peanut to the $2k+1$-th monkey.

   (a) How many monkeys will get no peanut?

   (b) How many sides of the polygon are there, for which both monkeys at its endpoints will get a peanut?

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