Vietnamese IMO Team Selection Test 1997

First Day – May 16

1. Let $ABCD$ be a given tetrahedron. Prove that there is a unique point P satisfying
$$\begin{aligned} & AP^2+AB^2+AC^2+AD^2= && BP^2+BA^2+BC^2+BD^2= \\ = \; & CP^2+CA^2+CB^2+CD^2= && DP^2+DA^2+DB^2+DC^2, \end{aligned}$$
and that for this point P we have $PA^2+PB^2+PC^2+PD^2 \geq 4R^2$, where R is the circumradius of the tetrahedron $ABCD$. Find the necessary and sufficient condition so that this inequality is an equality.

2. There are 25 towns in a country. Find the smallest k for which one can set up two-direction flight routes connecting these towns so that the following conditions are satisfied:
   (i) from each town there are exactly k direct routes to k other towns;
   (ii) if two towns are not connected by a direct route, then there is a town which has direct routes to these two towns.

3. Find the greatest real number α for which there exists a sequence $(a_n)_{n=1}^{\infty}$ of integers satisfying the following conditions:
   (i) $a_n > 1997^n$ for every $n \in \mathbb{N}$;
   (ii) $a_n^{\alpha} \leq U_n$ for every $n \geq 2$, where $U_n = \gcd\{a_i + a_j \mid i + j = n\}$.

Second Day – May 17

4. The function $f : \mathbb{N}_0 \to \mathbb{Z}$ is defined by $f(0) = 2$, $f(1) = 503$ and $f(n+2) = 503f(n+1) - 1996f(n)$ for all $n \geq 0$. Let $s_1, s_2, \dots, s_k$ be arbitrary integers not smaller than k, and let $p(s_i)$ be an arbitrary prime divisor of $f(2^{s_i})$ $(i = 1, \dots, k)$. Prove that, for any positive integer $t \leq k$,
$$\sum_{i=1}^{k} p(s_i) \mid 2^t \qquad \text{if and only if} \qquad k \mid 2^t.$$

5. Find all pairs of positive real numbers (a,b) such that for every $n \in \mathbb{N}$ and every real root x_n of the equation $4n^2x = \log_2(2n^2x+1)$ we have
$$a^{x_n} + b^{x_n} \geq 2 + 3x_n.$$

6. Let n,k,p be positive integers with $2 \leq k \leq \frac{n}{p+1}$. Let n distinct points on a circle be given. These points are colored blue and red so that exactly k points are blue and, on each arc determined by two consecutive blue points in clockwise direction, there are at least p red points. How many such colorings are there?

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com