Vietnamese IMO Team Selection Test 1985

First Day

1. The sequence (x_n) of real numbers is defined by $x_1 = 29/10$ and
$$x_{n+1} = \frac{x_n}{\sqrt{x_n^2 - 1}} + \sqrt{3}, \qquad \text{for } n \geq 1.$$
Find a real number a (if it exists) such that $x_{2k-1} > a > x_{2k}$ for all $k \in \mathbb{N}$.

2. Let ABC be a triangle with $AB = AC$. A ray Ax is constructed in space such that the three planar angles of the trihedral angle $ABCx$ at its vertex A are equal. If a point S moves on Ax, find the locus of the incenter of $\triangle SBC$.

3. Does there exist a triangle ABC satisfying the following two conditions:
   (i) $\sin^2 A + \sin^2 B + \sin^2 C = \cot A + \cot B + \cot C$;
   (ii) $S \geq a^2 - (b-c)^2$, where S is the area of the triangle?

Second Day

4. A convex polygon $A_1A_2\dots A_n$ is inscribed in a circle with center O and radius R so that O lies inside the polygon. Let the inradii of triangles $A_1A_2A_3, A_1A_3A_4, \dots, A_1A_{n-1}A_n$ be denoted by $r_1, r_2, \dots, r_{n-2}$. Show that
$$r_1 + r_2 + \dots + r_{n-2} \leq R\left(n\cos\frac{\pi}{n} - n + 2\right).$$

5. Find all real values of a for which the equation
$$\left(a - 3x^2 + \cos\frac{9\pi x}{2}\right)\sqrt{3 - ax} = 0$$
has an odd number of solutions in the interval $[-1, 5]$.

6. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(f(x)) = -x$ for every x. Prove that f has infinitely many points of discontinuity.

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