Vietnamese IMO Team Selection Test 2005

First Day

1. The incircle of a triangle ABC touches the sides BC, CA, AB at D, E, F, respectively. Circles $\omega_a, \omega_b, \omega_c$ are tangent to the incircle at D, E, F respectively and to the circumcircle of $\triangle ABC$ at K, M, N respectively.

   (a) Show that the lines DK, EM, FN pass through a single point P.

   (b) Prove that the orthocenter of $\triangle DEF$ lies on line OP.

2. There are n chairs at a round table, labelled with 1 through n. There are $k \le n/4$ students to take seats on these chairs. Any two neighboring students (i.e. those with no other students sitting between) must be separated by at least three chairs between them. Find the number of possible choices of the k chairs on which the students will be sitting.

3. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying
$$f(x^3 + y^3 + z^3) = f(x)^3 + f(y)^3 + f(z)^3 \quad \text{for all } x, y, z \in \mathbb{Z}.$$

Second Day

4. Let a, b, c be positive numbers. Prove the inequality
$$\frac{a^3}{(a+b)^3} + \frac{b^3}{(b+c)^3} + \frac{c^3}{(c+a)^3} \ge \frac{3}{8}.$$

5. Let $p > 3$ be a prime number. Calculate:

   (a) $S = \sum_{k=1}^{\frac{p-1}{2}} \left[\frac{2k^2}{p}\right] - 2\left[\frac{k^2}{p}\right]$ if $p \equiv 1 \pmod 4$;

   (b) $T = \sum_{k=1}^{\frac{p-1}{2}} \left[\frac{k^2}{p}\right]$ if $p \equiv 1 \pmod 8$.

6. A natural number n is called a *diamond-2005* if its decimal representation contains 2005 consecutive digits 9. If (a_n) is an arbitrary increasing sequence such that $a_n < Cn$ for each $n > 0$, where C is a constant, prove that (a_n) contains infinitely many diamonds-2005.

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