Vietnamese IMO Team Selection Test 2002

First Day

1. In a triangle ABC with an acute angle at C, let H be the feet of the altitude from A, M be the midpoint of BC, x and y be the trisectors of the angle A (where $\angle BAx = \angle xAy = \angle yAC$), and N, P be the intersection points of the perpendicular bisector of side BC with rays x and y, respectively. Find all triangles ABC with the property that $AB = NP = 2HM$.

2. A positive integer N_0 is written on a board. Two players A and B play the following game. A player on turn erases the number on the board (denoted by N) and writes one of the numbers $N-1$, $[N/3]$. They play alternately, and A begins. The player who first manages to write number 0 wins the game. Who has a winning strategy if N_0 is

$$\text{(a) } N_0 = 120; \quad \text{(b) } N_0 = \frac{3^{2002}-1}{2}; \quad \text{(c) } N_0 = \frac{3^{2002}+1}{2}\ ?$$

3. A positive integer m has a prime divisor greater than $\sqrt{2m}+1$. Find the least positive integer M for which there is a finite set T of positive integers satisfying the following conditions:

   (i) m is the least and M the greatest element in T;

   (ii) the product of all numbers in T is a perfect square.

Second Day

4. Consider a rectangular board $n \times 2n$, where $n > 2$ is a given integer. One marks n^2 arbitrary cells of the board. Prove that for any positive integer $k \le [n/2]+1$ there exist k rows of the board such that the rectangular board $k \times 2n$ formed by these k rows has at least

$$\frac{k!(n-2k+2)}{(n-1)(n-2)\cdots(n-k+1)}$$

   columns in which all cells are marked.

5. Find all polynomials $P(x)$ with integer coefficients for which there is a polynomial $Q(x)$ with integer coefficients such that

$$Q(x)^2 = (x^2+6x+10)P(x)^2 - 1.$$

6. Prove that there exist distinct positive integers $a_1, a_2, \ldots, a_m$ with $m \ge 2002$ such that $\prod_{i=1}^{m} a_i^2 - 4\sum_{i=1}^{m} a_i^2$ is a perfect square.

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