Vietnamese IMO Team Selection Test 2000

First Day

1. Two circles C_1 and C_2 intersect at points P and Q. Their common tangent, closer to P than to Q, touches C_1 at A and C_2 at B. The tangents to C_1 and C_2 at P meet the other circle at points $E \neq P$ and $F \neq P$, respectively. Let H and K be the points on the rays AF and BE respectively such that $AH = AP$ and $BK = BP$. Prove that A, H, Q, K, B lie on a circle.

2. Let k be a given positive integer. Define $x_1 = 1$ and, for each $n > 1$, set x_{n+1} to be the smallest positive integer not belonging to the set
$$\{x_i, x_i + ik \mid i = 1, \dots, n\}.$$
Prove that there is a real number a such that $x_n = [an]$ for all $n \in \mathbb{N}$.

3. Two players alternately replace the stars in the expression
$$*x^{2000} + *x^{1999} + \cdots + *x + 1$$
by real numbers. The player who makes the last move loses if the resulting polynomial has a real root t with $|t| < 1$, and wins otherwise. Give a winning strategy for one of the players.

Second Day

4. Let a, b, c be pairwise coprime natural numbers. A positive integer n is said to be *stubborn* if it cannot be written in the form
$$n = bcx + cay + abz, \quad \text{for some } x, y, z \in \mathbb{N}.$$
Determine the number of stubborn numbers.

5. Let $a > 1$ and $r > 1$ be real numbers.
   (a) Prove that if $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying the conditions
      (i) $f(x)^2 \leq ax^r f(x/a)$ for all $x > 0$,
      (ii) $f(x) < 2^{2000}$ for all $x < 1/2^{2000}$,
   then $f(x) \leq x^r a^{1-r}$ for all $x > 0$.
   (b) Construct a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying condition (i) such that for all $x > 0$, $f(x) > x^r a^{1-r}$.

6. A collection of 2000 congruent circles is given on the plane such that no two circles are tangent and each circle meets at least two other circles. Let N be the number of points that belong to at least two of the circles. Find the smallest possible value of N.

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