32-nd Vietnamese Mathematical Olympiad 1994

First Day – March 2

1. Solve the system of equations in x,y,z:

$$x^3+3x-3+\ln(x^2-x+1)=y;$$
$$y^3+3y-3+\ln(y^2-y+1)=z;$$
$$z^3+3z-3+\ln(z^2-z+1)=x.$$

2. Let ABC be a triangle and let A',B',C' be points symmetric to A,B,C with respect to BC,CA,AB, respectively. Find the necessary and sufficient condition for $\triangle ABC$ so that $\triangle A'B'C'$ is equilateral.

3. Given $0<a<1$, a sequence (x_n) is defined by $x_0=a$ and

$$x_n=\frac{4}{\pi^2}\left(\arccos x_{n-1}+\frac{\pi}{2}\right)\arcsin x_{n-1}, \qquad \text{for } n=1,2,\dots$$

Prove that this sequence converges and find its limit.

Second Day – March 3

4. Let be given a convex polygon $A_0A_1\dots A_n$ $(n>2)$. Initially, there are n stones at A_0. In each move we can choose two vertices A_i and A_j (not necessarily different), take a stone from each of them and put a stone onto a vertex adjacent to A_i and onto a vertex adjacent to A_j. Find all values of n such that after finitely many such moves we can achieve that each vertex except A_0 contains exactly one stone.

5. A sphere with center O and radius r is given. Two planes π and θ passing through O, perpendicular to each other, intersect the sphere in circles T_π and T_θ, respectively. Find the locus of the orthocenter H of an orthogonal tetrahedron $ABCD$ (i.e. with $AB\perp CD$, $AC\perp BD$ and $AD\perp BC$) with A on T_θ and B,C,D on T_π.

6. Do there exist polynomials $P(x),Q(x),T(x)$ with positive integer coefficients such that

$$T(x)=(x^2-3x+3)P(x),$$
$$P(x)=\left(\frac{1}{20}x^2-\frac{1}{15}x+\frac{1}{12}\right)Q(x)?$$

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