30-th Vietnamese Mathematical Olympiad 1992

First Day

1. A tetrahedron $ABCD$ is such that
$$\angle ACD + \angle BCD = \angle BAC + \angle CAD + \angle DAB =$$
$$= \angle ABC + \angle CBD + \angle DBA = 180^\circ.$$
Given that $\angle ACB = \alpha$ and $AC + CB = k$, compute the total area of the surface of the tetrahedron $ABCD$.

2. For a positive number n, $f(n)$ is the number of divisors of n which are congruent to 1 or -1 modulo 10, and $g(n)$ is the number of divisors which are congruent to 3 or -3 modulo 10. Prove that $f(n) \geq g(n)$.

3. Three real sequences (a_n), (b_n), (c_n) are constructed as follows:

   (i) $a_0 = a$, $b_0 = b$, $c_0 = c$, where a, b, c are given real numbers

   (ii) $a_{k+1} = a_k + \dfrac{2}{b_k + c_k}$, $b_{k+1} = b_k + \dfrac{2}{c_k + a_k}$, $c_{k+1} = c_k + \dfrac{2}{a_k + b_k}$ for all k.

   Prove that a_n tends to infinity as n tends to infinity.

Second Day

4. The field of a 1991×1992 board in the m-th row and n-th column is denoted as (m, n). We color some squares of the board as follows. At first, we color fields (r, s), $(r+1, s+1)$ and $(r+2, s+1)$, where r, s are given numbers with $1 \leq r \leq 1989$ and $1 \leq s \leq 1991$. Afterwards, at each step we color red three yet uncolored fields which are in the same row or column. Can we color all the fields of the board according to this rule?

5. The two diagonals of a rectangle $\mathscr{H}$ form an angle not exceeding 45°. The rectangle $\mathscr{H}$, when rotated around its center for an angle x $(0 \leq x < 360^\circ)$, maps onto a triangle $\mathscr{H}_x$. Find x for which the area of the intersection of $\mathscr{H}$ and $\mathscr{H}_x$ is the greatest possible.

6. Let $n_1 < n_2 < \cdots < n_k$ be positive integers. Prove that all real roots of the polynomial $P(x) = 1 + x^{n_1} + x^{n_2} + \cdots + x^{n_k}$ are greater than $\dfrac{1-\sqrt{5}}{2}$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com