29-th Vietnamese Mathematical Olympiad 1991

First Day

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\frac{f(xy)+f(xz)}{2} - f(x)f(yz) \geq \frac{1}{4} \quad \text{for all } x,y,z \in \mathbb{R}.$$

2. Let $k > 1$ be an odd integer. For every positive integer n, let $f(n)$ be the greatest positive integer for which $2^{f(n)}$ divides $k^n - 1$. Find $f(n)$ in terms of k and n.

3. Three mutually perpendicular rays Ox, Oy, Oz and three points A, B, C on Ox, Oy, Oz, respectively. A variable sphere ε through A, B, C meets Ox, Oy, Oz again at A', B', C', respectively. Let M and M' be the centroids of triangles ABC and $A'B'C'$. Find the locus of the midpoint of MM'.

Second Day

4. 1991 students sit around a circle and play the following game. Starting from some student A and counting clockwise, each student on turn says a number. The numbers are $1,2,3,1,2,3,\dots$. A student who says 2 or 3 must leave the circle. The game is over when there is only one student left. What position was the remaining student sitting at the beginning of the game?

5. Let G be the centroid and R the circumradius of a triangle ABC. The extensions of GA, GB, GC meet the circumcircle again at D, E, F. Prove that

$$\frac{3}{R} \leq \frac{1}{GD} + \frac{1}{GE} + \frac{1}{GF} \leq \sqrt{3} \leq \left(\frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA}\right).$$

6. If $x \geq y \geq z \geq 0$ are real numbers, prove that

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2.$$

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