28-th Vietnamese Mathematical Olympiad 1990

First Day

1. The sequence (x_n) is defined by $x_1 = a$, $|a| < 1$, and for all $n \geq 1$,
$$x_{n+1} = \frac{-x_n + \sqrt{3 - 3x_n^2}}{2}.$$
   (a) Find the necessary and sufficient condition for a so that each $x_n > 0$.
   (b) Prove that the sequence is periodic for any choice of a.

2. At least $n-1$ numbers are removed from the set $A = \{1, 2, \dots, 2n-1\}$ according to the following rules:
   (i) If a is removed, so is $2a$;
   (ii) If a and b are removed, so is $a+b$.

   Find the way of removing numbers such that the sum of the remaining numbers is maximum possible.

3. A tetrahedron is to be cut by three planes which form a parallelepiped whose three faces and all vertices lie on the surface of the tetrahedron.
   (a) Can this be done so that the volume of the parallelepiped is at least $9/40$ of the volume of the tetrahedron?
   (b) Determine the common point of the three planes if the volume of the parallelepiped is $11/50$ of the volume of the tetrahedron.

Second Day

4. A triangle ABC is given in the plane. Let M be a point inside the triangle and A', B', C' be its projections on the sides BC, CA, AB, respectively. Find the locus of M for which $MA \cdot MA' = MB \cdot MB' = MC \cdot MC'$.

5. Suppose $P(x)$ is a non-constant polynomial with real coefficients satisfying
$$P(x)P(2x^2) = P(3x^3 + x) \qquad \text{for all } x \in \mathbb{R}.$$
   Prove that P has no real roots.

6. The children sitting around a circle are playing the game as follows. At first the teacher gives each child an even number of candies (may be 0). A certain child gives half of his candies to his neighbor on the right. Then the child who has just received candies does the same if he has an even number of candies, otherwise he gets one candy from the teacher and then does the job; and so on.

   Prove that after several steps there will be a child who will be able, giving the teacher half of his candies, to make the numbers of candies of all the children equal.

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