25-th Vietnamese Mathematical Olympiad 1987

First Day

1. Let $u_1, u_2, \dots, u_{1987}$ be an arithmetic progression with $u_1 = \pi/1987$ and the common difference $\pi/3974$. Evaluate
$$S = \sum_{\varepsilon_i \in \{-1,1\}} \cos(\varepsilon_1 u_1 + \varepsilon_2 u_2 + \dots + \varepsilon_{1987} u_{1987}).$$

2. Sequences (x_n) and (y_n) are constructed as follows:
$$\begin{array}{ll} x_0 = 365, & x_{n+1} = x_n(x_n^{1986}+1)+1622, \\ y_0 = 16, & y_{n+1} = y_n(y_n^3+1)-1952, \qquad \text{for all } n \geq 0. \end{array}$$
Prove that $x_n \neq y_k$ for any positive integers n, k.

3. Let be given $n \geq 2$ lines on a plane, not all concurrent and no two parallel. Prove that there is a point which belongs to exactly two of the given lines.

Second Day

4. Let $a_1, a_2, \dots, a_n$ be positive real numbers ($n \geq 2$) whose sum is S. Show that
$$\sum_{i=1}^{n} \frac{a_i^{2^k}}{(S-a_i)^{2^t-1}} \geq \frac{S^{1+2^k-2^t}}{(n-1)^{2^t-1} n^{2^k-2^t}}$$
for any nonnegative integers k, t with $k \geq t$. When does equality occur?

5. Let $f : [0, +\infty) \to \mathbb{R}$ be a differentiable function. Suppose that
$$|f(x)| \leq 5 \quad \text{and} \quad f(x)f'(x) \geq \sin x \quad \text{for all } x \geq 0.$$
Prove that there exists $\lim_{x\to+\infty} f(x)$.

6. Prove that among any five distinct rays Ox, Oy, Oz, Ot, Or in space there exist two which form an angle less than or equal to 90°.

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