47-th Vietnamese Mathematical Olympiad 2009

First Day

1. Solve the system of equations:

$$\begin{cases} \dfrac{1}{\sqrt{1+2x^2}} + \dfrac{1}{\sqrt{1+2y^2}} = \dfrac{2}{\sqrt{1+2xy}} \\ \sqrt{x(1-2x)} + \sqrt{y(1-2y)} = \dfrac{2}{9} \end{cases}$$

2. The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 = \frac{1}{2} \\ x_n = \frac{\sqrt{x_{n-1}^2 + 4x_{n-1}} + x_{n-1}}{2} \end{cases}$$

   Prove that the sequence $\{y_n\}$, where $y_n = \sum_{i=1}^{n} \frac{1}{x_i{}^2}$, has a finite limit and find that limit.

3. Let A, B be two fixed points and C a variable point in a plane such that $\angle ACB = \alpha$ is a constant ($0^\circ \le \alpha \le 180^\circ$). Let D, E, F be the projections of the incenter I of triangle ABC to its sides BC, CA, AB, respectively. Denote by M, N the intersections of AI, BI with EF, respectively. Prove that the length of the segment MN is constant and the circumcircle of triangle DMN always passes through a fixed point.

4. Let a, b, c be three real numbers. For each positive integer number n, $a^n + b^n + c^n$ is an integer number. Prove that there exist three integers p, q, r such that a, b, c are the roots of the equation $x^3 + px^2 + qx + r = 0$.

5. Let $S = \{1, 2, 3, \ldots, 2n\}$ ($n \in \mathbb{Z}^+$). Determine the number of subsets T of S such that there are no two elements a, b in T such that $|a-b| = \{1, n\}$.

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