45-th Vietnamese Mathematical Olympiad 2007

February 8
Time allowed: 3 hours

1. Solve the system of equations
$$1-\frac{12}{y+3x}=\frac{2}{\sqrt{x}},\quad 1+\frac{12}{y+3x}=\frac{6}{\sqrt{y}}.$$

2. Let x,y be integers different from -1 such that $\dfrac{x^4-1}{y+1}+\dfrac{y^4-1}{x+1}$ is an integer. Prove that $x^4y^{44}-1$ is divisible by $x+1$.

3. Points B and C in the plane are fixed, while point A varies. Let H be the orthocenter and G be the centroid of triangle ABC. Find the locus of points A for which the midpoint K of GH lies on line BC.

4. Consider a regular 2007-gon. Find the smallest k with the following property: In every set of k vertices there are four which form a quadrilateral three of whose sides are also sides of the 2007-gon.

5. Given a positive number b, find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying
$$f(x+y)=f(x)\cdot 3^{b^y+f(y)-1}+b^x\left(3^{b^y+f(y)-1}-b^y\right)\quad \text{for all } x,y.$$

6. A trapezoid $ABCD$ with $BC\parallel AD$ and $BC>AD$ is inscribed in a circle k with center O. A variable point P on the line BC outside the segment BC is such that PA does not touch k. The circle with diameter PD intersects k at $E\neq D$. The lines BC and DE meet at M, and PA intersects k again at N. Prove that the line MN passes through a fixed point.

7. Let $a>2$ be a given real number. For $n\in\mathbb{N}$ define
$$f_n(x)=a^{10}x^{n+10}+x^n+\cdots+x+1.$$
Prove that for every n the equation $f_n(x)=a$ has exactly one positive real root x_n. Prove that the sequence (x_n) has a finite limit when $n\to\infty$.

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