44-th Vietnamese Mathematical Olympiad 2006

First Day

1. Find all real solutions of the system:

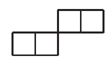
$$\begin{aligned}
\sqrt{x^2-2x+6}\cdot\log_3(6-y) &= x,\\
\sqrt{y^2-2y+6}\cdot\log_3(6-z) &= y,\\
\sqrt{z^2-2z+6}\cdot\log_3(6-x) &= z.
\end{aligned}$$

2. Let $ABCD$ be a convex quadrilateral and M be an arbitrary point on line AB. The circumcircles of triangles MAD and MBC meet again at N.

   (a) Prove that N lies on a fixed circle.

   (b) Show that line MN passes through a fixed point.

3. We are putting marbles onto the squares of an $m\times n$ board $(m,n>3)$ according to the following rule. In each step, a marble is put onto each of some four squares that form a figure congruent to the one shown on the image. Is it possible that after finitely many steps each square contains the same number of marbles, if



   (a) $(m,n)=(2004,2006)$?

   (b) $(m,n)=(2005,2006)$?

Second Day

4. Consider the function $f(x)=-x+\sqrt{(x+a)(x+b)}$, where a,b are given distinct positive numbers. Show that for each s with $0<s<1$ there is exactly one $\alpha>0$ such that

$$f(\alpha)=\sqrt[s]{\frac{a^s+b^s}{2}}.$$

5. Find all real polynomials $P(x)$ that satisfy the equality

$$P(x^2)+x\,(3P(x)+P(-x))=P(x)^2+2x^2 \quad \text{for all } x.$$

6. A set T is called *naughty* if for any two (not necessarily distinct) elements u,v of T, $u+v\notin T$. Prove that

   (a) a naughty subset of $S=\{1,2,\dots,2006\}$ has at most 1003 elements;

   (b) every set S of 2006 positive numbers contains a naughty subset having 669 elements.

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