# 43-rd Vietnamese Mathematical Olympiad 2005

*First Day*

1. The real numbers $x$ and $y$ satisfy the condition

$$x - 3\sqrt{x+1} = 3\sqrt{y+2} - y.$$

Find the maximum and minimum values of $P = x + y$.

2. Points $A$ and $B$ are given on a circle with center $O$ and radius $R$ so that $O$ does not lie on $AB$. Let $C$ be a variable point on the circle, distinct from $A$ and $B$. Circle $O_1$ through $A$ touches $BC$ at $C$, and circle $O_2$ through $B$ touches $AC$ at $C$. Circles $O_1$ and $O_2$ meet at $C$ and $D$.

   (a) Prove that $CD \leq R$.

   (b) Prove that all lines $CD$ pass through a fixed point.

3. Let $A_1A_2 \ldots A_8$ be a convex octagon with no three diagonals concurrent. We call intersections of its diagonals *buttons*, and convex quadrilaterals determined by four vertices of the octagon *sub-quadrilaterals*. Find the smallest $n$ for which it is possible to color $n$ buttons so that for all distinct $i,k \in \{1,\ldots,8\}$ number $s(i,k)$ is the same, where $s(i,k)$ denotes the number of sub-quadrilaterals with two vertices at $A_i, A_k$ whose diagonals meet at a colored button.

*Second Day*

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$f(f(x-y)) = f(x)f(y) - f(x) + f(y) - xy \quad \text{for all } x, y.$$

5. Find all triples of natural numbers $(x,y,n)$ such that $\dfrac{x!+y!}{n!} = 3^n$.

6. The sequence $(x_n)$ is defined by $x_1 = a \in \mathbb{R}$ and

$$x_{n+1} = 3x_n^3 - 7x_n^2 + 5x_n \quad \text{for } n = 1,2,3,\ldots$$

Find all $a$ for which $(x_n)$ has a finite limit when $n \to \infty$ and determine that limit.



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