39-th Vietnamese Mathematical Olympiad 2001

First Day – March 12

1. Two circles (O_1) and (O_2) intersect at A and B, and their common tangent touches (O_1) at P_1 and (O_2) at P_2. Let M_1 and M_2 be the orthogonal projections of P_1 and P_2 on the line O_1O_2. The line AM_1 cuts (O_1) again at N_1, and AM_2 cuts (O_2) again at N_2. Prove that the points B,N_1,N_2 are collinear.

2. Let be given a positive integer n and two coprime integers a,b greater than 1. Let p and q be two odd divisors of $a^{6^n}+b^{6^n}$ different from 1. Find the remainder of $p^{6^n}+q^{6^n}$ when divided by $6\cdot 12^n$.

3. Given real numbers a,b, the sequence $(x_n)_{n=0}^{\infty}$ is defined by $x_0=a$ and
$$x_{n+1}=x_n+b\sin x_n \quad \text{for every } n\geq 0.$$
   (a) If $b=1$, prove that for every a, the sequence (x_n) has a finite limit when $n\to\infty$, and find this limit.

   (b) Prove that for every $b>2$ there exists a real number a for which the sequence (x_n) does not have a finite limit when $n\to\infty$.

Second Day – March 13

4. Let x,y,z be positive real numbers that satisfy:
$$\frac{1}{\sqrt{2}}\leq z<\frac{1}{2}\min\{x\sqrt{2},y\sqrt{3}\};$$
$$x+z\sqrt{3}\geq\sqrt{6};$$
$$y\sqrt{3}+z\sqrt{10}\geq 2\sqrt{5}.$$
   Find the maximum value of $P(x,y,z)=\dfrac{1}{x^2}+\dfrac{2}{y^2}+\dfrac{3}{z^2}$.

5. Consider the function $g(x)=\dfrac{2x}{1+x^2}$. Find all continuous functions $f:(-1,1)\to\mathbb{R}$ that satisfy
$$(1-x^2)f(g(x))=(1+x^2)^2f(x) \quad \text{for all } x\in(-1,1).$$

6. Let $n\geq 1$ be a given integer. Consider a permutation $(a_1,a_2,\dots,a_{2n})$ of the first $2n$ positive integers such that the numbers $|a_{i+1}-a_i|$ are distinct for $i=1,2,\dots,2n-1$. Prove that $a_1-a_{2n}=n$ if and only if $1\leq a_{2k}\leq n$ for every $k=1,2,\dots,n$.

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