The Ukrainian Team Selection Tests 1999

April 20–29

First Test

1. A triangle ABC is given. Points E,F,G are arbitrarily selected on the sides AB,BC,CA, respectively, such that $AF \perp EG$ and the quadrilateral $AEFG$ is cyclic. Find the locus of the intersection point of AF and EG.

2. Show that there exist integers j,k,l,m,n greater than 100 such that
$$j^2+k^2+l^2+m^2+n^2 = jklmn-12.$$

3. Let m,n be positive integers with $m \le n$, and let $\mathscr{F}$ be a family of m-element subsets of $\{1,2,\dots,n\}$ satisfying $A \cap B \neq \emptyset$ for all $A,B \in \mathscr{F}$. Determine the maximum possible number of elements in $\mathscr{F}$.

Second Test

1. If $n \in \mathbb{N}$ and $0 < x < \frac{\pi}{2n}$, prove the inequality
$$\frac{\sin 2x}{\sin x} + \frac{\sin 3x}{\sin 2x} + \cdots \frac{\sin(n+1)x}{\sin nx} < 2\frac{\cos x}{\sin^2 x}.$$

2. A convex pentagon $ABCDE$ with $DC = DE$ and $\angle DCB = \angle DEA = 90^\circ$ is given. Let F be a point on the segment AB such that $AF : BF = AE : BC$. Prove that
$$\angle FCE = \angle ADE \quad \text{and} \quad \angle FEC = \angle BDC.$$

3. Show that for any $n \in \mathbb{N}$ the polynomial $f(x) = (x^2+x)^{2^n}+1$ is irreducible over $\mathbb{Z}[x]$.

Third Test

1. Let $P_1P_2\dots P_n$ be an oriented closed polygonal line with no three segments passing through a single point. Each point P_i is assinged the angle $180^\circ - \angle P_{i-1}P_iP_{i+1} \ge 0$ if P_{i+1} lies on the left from the ray $P_{i-1}P_i$, and the angle $-(180^\circ - \angle P_{i-1}P_iP_{i+1}) < 0$ if P_{i+1} lies on the right. Prove that if the sum of all the assigned angles is a multiple of 720°, then the number of self-intersections of the polygonal line is odd.

2. Find all pairs (x,n) of positive integers for which $x^n + 2^n + 1$ divides $x^{n+1} + 2^{n+1} + 1$.

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3. Find all functions $u : \mathbb{R} \to \mathbb{R}$ for which there is a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x)u(y) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Fourth Test

1. For a natural number n, let $w(n)$ denote the number of (positive) prime divisors of n. Find the smallest positive integer k such that

$$2^{w(n)} \leq k\sqrt[4]{n} \quad \text{for each } n \in \mathbb{N}.$$

2. Let $ABCDEF$ be a convex hexagon such that $BCEF$ is a parallelogram and ABF an equilateral triangle. Given that $BC = 1$, $AD = 3$, $CD + DE = 2$, compute the area of $ABCDEF$.

3. In a group of $n \geq 4$ persons, every three who know each other have a common signal. Assume that these signals are not repeated and that there are $m \geq 1$ signals in total. For any set of four persons in which there are three having a common signal, the fourth person has a common signal with at most one of them. Show that there three persons who have a common signal, such that the number of persons having no signal with anyone of them does not exceed $\left[n + 3 - \dfrac{18m}{n}\right]$.