39-th All-Ukrainian Mathematical Olympiad 1999

Final Round – Zaporizhya, March 14–20

Grade 8

First Day

1. Solve the system of equations $2|x|+|y|=1$, $[|x|]+[2|y|]=2$.

2. Is it possible to write numbers in the cells of a 7×7 board in such a way that the sum of numbers in every 2×2 or 3×3 square is divisible by 1999, but the sum of all numbers in the board is not divisible by 1999?

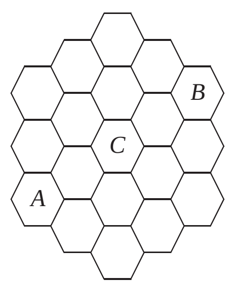
3. Is there a 2000-digit number which is a perfect square and 1999 of whose digits are fives.

Second Day

4. Can the number 19991999 be written in the form n^4+m^3-m, where n,m are integers?

5. Let N be the point inside a rhombus $ABCD$ such that the triangle BNC is equilateral. The bisector of $\angle ABN$ meets the diagonal AC at K. Show that $BK=KN+ND$.

6. Consider the figure consisting of 19 hexagonal cells, as shown on the picture. At the cell A there is a piece which is allowed to move one cell up, up-right, or down-right. How many ways are there for the piece to reach the cell B, not passing through the cell C?



Grade 9

First Day

1. Describe the region in the coordinate plane defined by $|x^2+xy|\geq|x^2-xy|$.

2. Let x and y be positive real numbers with $(x-1)(y-1)\geq 1$. Prove that for sides a,b,c of an arbitrary triangle we have $a^2x+b^2y>c^2$.

3. Show that the number $9999999+1999000$ is composite.

4. The bisectors of angles A, B, C of a triangle ABC intersect the circumcircle of the triangle at A_1, B_1, C_1, respectively. Let P be the intersection of the lines B_1C_1 and AB, and Q be the intersection of the lines B_1A_1 and BC. Show how to construct the triangle ABC by a ruler and a compass, given its circumcircle, points P and Q, and the halfplane determined by PQ in which point B lies.

Second Day

5. Solve the equation $[x] + \dfrac{1999}{[x]} = \{x\} + \dfrac{1999}{\{x\}}$.

6. Find all pairs (k, l) of positive integers such that $\dfrac{k^l}{l^k} = \dfrac{k!}{l!}$.

7. Let M be a fixed point inside a given circle. Two perpendicular chords AC and BD are drawn through M, and K and L are the midpoints of AB and CD, respectively. Prove that the quantity $AB^2 + CD^2 - 2KL^2$ is independent of the chords AC and BD.

8. A sequence of natural numbers (a_n) satisfies $a_{a_n} + a_n = 2n$ for all $n \in \mathbb{N}$. Prove that $a_n = n$.

Grade 10

First Day

1. Solve the equation $\sin x \sin 2x \sin 3x + \cos x \cos 2x \cos 3x = 1$.

2. Let M be a point inside a triangle ABC. The line through M parallel to AC meets AB at N and BC at K. The lines through M parallel to AB and BC meet AC at D and L, respectively. Another line through M intersects the sides AB and BC at P and R respectively such that $PM = MR$. Given that the area of $\triangle ABC$ is S and that $CK/CB = a$, compute the area of $\triangle PQR$.

3. Let $P(x)$ be a polynomial with integer coefficients. Suppose that a sequence (x_n) of integers satisfies $x_1 = x_{2000} = 1999$ and $x_{n+1} = x_n$ for all $n \in \mathbb{N}$. Determine

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{1999}}{a_{2000}}.$$

4. Two players alternately write integers on a blackboard as follows: the first player writes a_1 arbitrarily, then the second player writes a_2 arbitrarily, and thereafter a player writes a number that is equal to the sum of the two preceding numbers. The player after whose move the obtained sequence contains terms such that $a_i - a_j$ and $a_{i+1} - a_{j+1}$ $(i \neq j)$ are divisible by 1999, wins the game. Which of the players has a winning strategy?

Second Day

5. Evaluate
$$[\pi]+\left[\frac{[2\pi]}{2}\right]+\left[\frac{[3\pi]}{3}\right]+\cdots+\left[\frac{[1999\pi]}{1999}\right].$$

6. Solve the equation $m^3-n^3=7mn+5$ in positive integers.

7. Let $x_1,x_2,\dots,x_6$ be numbers from the interval $[0,1]$. Prove that
$$\frac{x_1^3}{x_2^5+\cdots+x_6^5+5}+\cdots+\frac{x_6^3}{x_1^5+\cdots+x_5^5+5}\le\frac{3}{5}.$$

8. Let AA_1,BB_1,CC_1 be the altitudes of an acute-angled triangle ABC, and let O be an arbitrary interior point. Let M,N,P,Q,R,S be the feet of the perpendiculars from O to the lines AA_1,BC,BB_1,CA,CC_1,AB, respectively. Prove that the lines MN,PQ,RS are concurrent.

Grade 11

First Day

1. Solve the equation
$$(\sin x)^{1998}+(\cos x)^{-1999}=(\cos x)^{1998}+(\sin x)^{-1999}.$$

2. Find all values of the parameter k for which the system of inequalities
$$\begin{aligned}ky^2+4ky-2x+6k+3&\le 0\\ kx^2-2y-2kx+3k-3&\le 0\end{aligned}$$
has a unique solution.

3. All faces of a parallelepiped $ABCDA_1B_1C_1D_1$ are rhombi, and their angles at A are all equal to α. Points M,N,P,Q are selected on the edges A_1B_1,DC,BC,A_1D_1, respectively, such that $A_1M=BP$ and $DN=A_1Q$. Find the angle between the intersection lines of the plane A_1BD with the planes AMN and APQ.

4. *Problem 4 for Grade 10.*

Second Day

5. Can the number (a) 19991998, (b) 19991999 be written in the form n^4+m^3-m, where n,m are integers?

6. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
$$f(xy)+f(xz)-f(x)f(yz)\ge 1\quad\text{for all } x,y,z.$$

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7. Suppose that the function $f(x) = \tan(a_1x+1) + \cdots + \tan(a_{10}x+1)$ has the period $T > 0$, where $a_1, \ldots, a_{10}$ are positive numbers. Prove that

$$T \geq \frac{\pi}{10} \min\left\{\frac{1}{a_1}, \ldots, \frac{1}{a_{10}}\right\}.$$

8. *Problem 8 for Grade 10.*