38-th All-Ukrainian Mathematical Olympiad 1998

Final Round – Mykolaiv, March 9–10

Grade 8

First Day

1. In a family of five members (father, mother and three children), the product of the (integral) ages of all members equals 1998. Knowing that father is 10 years older than mother, determine the ages of all members.

2. The plane is partitioned into congruent regular hexagons. Of these hexagons, some 1998 are marked. Show that one can select 666 of the marked hexagons in such a way that no two of them share a vertex.

3. On the lower-left corner of an $n \times n$ board there is a piece. In a move a player can move the piece to the adjacent cell up, right, or up-right. Two players alternate moving the piece, and the winner is the one who places the piece on the upper-right corner. Who has a winning strategy?

Second Day

4. Determine the last four decimal digits of the number $1997 \cdot 5^{1998}$.

5. A line l and points A, B on the same side of l are given in the plane. Construct (with a ruler and a compass) a point C such that the line l intersects AC at M and BC at N, where BM is the altitude and AN the median.

6. Show that it is possible to write a number from the set $\{1,2,3,4,5\}$ in each square of a 5×120 board (5 columns and 120 rows) so that the following conditions are satisfied:

   (i) all numbers in the same row are distinct;

   (ii) all rows are distinct;

   (iii) the board can be partitioned into 24 5×5 boards which can be reassembled (without rotating and turning over) into a 120×5 board whose 120 columns are distinct.

Grade 9

First Day

1. Is there a triangle in the coordinate plane whose vertices, centroid, orthocenter, incenter and circumcenter all have integral coordinates?

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2. A quadrilateral $ABCD$ is inscribed in a circle with diameter AD. Using a ruler and a compass, construct a triangle inscribed in the same circle and having the same area as $ABCD$.

3. Prove that for any positive numbers a,b,c satisfying $abc = 1$
$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ca}{1+c} \geq 3.$$

4. Real numbers x and y are not less than 1 and have the property that
$$\left[\frac{x}{y}\right] = \frac{[nx]}{[ny]} \quad \text{for any } n \in \mathbb{N}.$$
Prove that either $x = y$ or x and y are integers, one dividing the other.

Second Day

5. Prove that the equation $x^3 - x = y^2 - 19y + 98$ has no solutions in integers.

6. Prove that the sum of squared lengths of the medians of a triangle does not exceed the square of its semiperimeter.

7. Two players alternately write numbers in the cells of an $n \times n$ square board. Hereby, in the intersection of the i-th row and the j-th column the first player may write the greatest common divisor of numbers i and j, whereas the second player may write their least common multiple. When the board is filled up, the numbers of the first column are divided by 1, those of the second column by 2, etc, those of the last column by n. Then the product of the obtained numbers in the board is computed. If the result is smaller than 1, the first player wins, otherwise the second player wins. Which player has a winning strategy?

8. A convex 2000-gon is given in the plane. Show that one can select 1998 points in the plane such that every triangle with the vertices at vertices of the 2000-gon contains exactly one of the selected points in its interior (excluding the boundary).

Grade 10

First Day

1. Find all pairs of real numbers (x,y) satisfying the equation
$$\sin^2 x + \sin^2 y = \frac{|x|}{x} + \frac{|y|}{y}.$$

2. Point M is arbitrarily taken on side AC of a triangle ABC. Let O be the intersection of the perpendiculars from the midpoints of segments AM and MC to lines BC and AB, respectively. For which position of M is the distance OM minimal?

3. A finite set of segments on a line has the following property: In any subset of 1998 segments there are two having a common point. Show that there exist 1997 points on the line such that each segment contains at least one of these points.

4. A real function f defined on the interval $[1,+\infty)$ satisfies
$$\frac{f(2x)}{\sqrt{2}} \le f(x) \le x \quad \text{for } x \ge 1.$$
Prove that $f(x) < \sqrt{2x}$ for all $x \ge 1$.

Second Day

5. Let m be the smallest of the four numbers $1, x^9, y^9, z^7$, where x,y,z are nonnegative numbers. Prove that $m \le xy^2z^3$.

6. Let AB and CD be diameters of a circle with center O. For a point M on a shorter arc CB, lines MA and MD meet the chord BC at points P and Q respectively. Prove that the sum of the areas of the triangles CPM and MQB equals the area of triangle DPQ.

7. Baron Munchausen claims that he can accomodate an arbitrary set of guests in the rooms of his castle in such a way that the guests in each room either all know each other or all do not know each other. Is his claim true? (Assume that the rooms are large enough to accomodate any number of people, but that there are finitely many rooms.)

8. Find all pairs of polynomials $f(x), g(x)$ that satisfy $f(xy) = f(x) + g(x)f(y)$ for all x,y.

Grade 11

First Day

1. Solve the equation $\sqrt{1+\{2x\}} = [x^2] + 2[x] + 3$ (where $\{a\} = a - [a]$).

2. The altitude CD of triangle ABC meets the bisector BK of this triangle at M and the altitude KL of $\triangle BKC$ at N. The circumcircle of triangle BKN meets the side AB at point $P \neq B$. Prove that the triangle KPM is isosceles.

3. For any numbers $0 < x,y,z \le 1$ prove the inequality
$$\frac{x}{1+y+zx} + \frac{y}{1+z+xy} + \frac{z}{1+x+yz} \le \frac{3}{x+y+z}.$$

4. Consider a function $f : [0,1] \to [0,1]$. Suppose that there is a real number $0 < \lambda < 1$ such that $f(\lambda) \notin \{0, \lambda\}$ and the equality
$$f(f(x)+y) = f(x) + f(y)$$
holds whenever the function is defined on the arguments.

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(a) Give an example of such a function.

(b) Prove that for some $x \in [0,1]$,

$$\underbrace{f(f(\cdots f}_{19}(x)\cdots)) = \underbrace{f(f(\cdots f}_{98}(x)\cdots)).$$

Second Day

5. Find the number of real roots of the equation

$$\frac{2}{\pi}\arccos x = \sqrt{1-x^2}+1.$$

6. The numbers $1, 1998, 1999$ are written on the board. In each step it is allowed to replace one of the numbers by its square decreased by three times the product of the other two numbers. Is it possible to obtain three numbers with the sum 0 after several steps?

7. Two spheres are externaly tangent at point P. The segments AB and CD touch the spheres with A and C lying on the first sphere and B and D on the second. Let M and N be the projections of the midpoints of segments AC and BD on the line connecting the centers of the spheres. Prove that $PM = PN$.

8. The sequence (x_n) is given by $x_1 = 1$ and

$$x_{n+1} = \frac{n^2}{x_n} + \frac{x_n}{n^2} + 2 \quad \text{for } n \geq 1.$$

(a) Prove that $x_{n+1} \geq x_n$ for all $n \geq 4$.

(b) Prove that $[x_n] = n$ for all $n \geq 4$.

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