37-th All-Ukrainian Mathematical Olympiad 1997

Final Round – Odessa, February 26–27

Grade 8

First Day

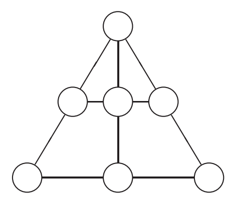
1. Which number is greater: $4^{4^{4^{4^4}}}$ or $5^{5^{5^5}}$?

2. There are n candies on a table. Petrik and Mikola alternately take candies from the table according to the following rule. Petrik starts by taking one candy; then Mikola takes i candies, where i divides 2, then Petrik takes j candies, where j divides 3, and so on. The player who takes the last candy wins the game. Which player has a winning strategy?

3. Consider acute-angled triangles ABC and APQ, where P and Q lie on the side BC. Prove that the circumcenter of $\triangle ABC$ is closer to line BC than the circumcenter of $\triangle APQ$.

Second Day

4. Two players alternately write the numbers $1,2,\dots,7$ in the cells of the figure on the picture. After the figure is filled up, the sum of numbers along each line on the picture is calculated. If three of these sums are equal, the first player wins; otherwise the second player wins. Which of the players has a winning strategy?



5. Construct the bisector of a given angle using a ruler and a compass, but without marking any auxilliary points inside the angle.

6. Find all natural numbers n for which $2^n - n^2$ is divisible by 7.

Grade 9

First Day

1. There are four apples on a plate, of weights $600, 400, 300$, and 250 grammes. Two pupils take an apple each (one by one) and start eating simultaneously. A pupil can take another apple only after he has eaten the previous apple. It is assumed that they eat equaly fast (measured in grammes per second). How should the first pupil do in order to eat as large amount as possible?

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

2. Solve in real numbers the equation $9^x + 4^x + 1 = 6^x + 3^x + 2^x$.

3. The incircle of a triangle ABC is tangent to its sides AB, BC, CA at M, N, K, respectively. A line l through the midpoint D of AC is parallel to MN and intersects the lines BC and BD at T and S, respectively. Prove that $TC = KD = AS$.

4. Does there exist a positive integer a such that all the numbers $a, 2a, 3a, \dots, 1997a$ are perfect powers (i.e. numbers of the form m^k, where $m, k \in \mathbb{N}$, $k \geq 2$)?

Second Day

5. Find the smallest natural number which can be represented in the form $19m + 97n$, where m, n are positive integers, in at least two different ways.

6. The cells of a rectangular board are colored in the chess manner. Every cell is filled with an integer. Assume that the sum of numbers in any row and in any column is even. Prove that the sum of numbers in the black cells is even.

7. Triangles ABC and $A_1B_1C_1$ are non-congruent, but $AC = A_1C_1 = b$, $BC = B_1C_1 = a$, and $BH = B_1H_1$, where BH and B_1H_1 are the altitudes. Prove the inequality
$$a \cdot AB + b \cdot A_1B_1 \leq \sqrt{2}(a^2 + b^2).$$

8. For every natural number n prove the inequality
$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots + \sqrt{6}}}} < 5,$$
where there are n twos and n sixs on the left hand side.

Grade 10

First Day

1. Find the maximum value of the function
$$f(x) = \frac{x}{x^2 + 9} + \frac{1}{x^2 - 6x + 21} + \cos 2\pi x, \quad x > 0.$$

2. Each side of an equilateral triangle is divided into 1997 equal parts, and each vertex of the triangle is joined by a segment to each of the division points on the opposite side. All the intersection points of these segments inside the triangle are marked. How many marked points are there?

3. A sequence is defined by $a_1 = a_2 = a_3 = 1$ and $a_{n+3} = -a_n - a_{n+1}$ for all $n \in \mathbb{N}$. Prove that this sequence is not bounded, i.e. that for every $M \in \mathbb{R}$ there is an n for which $|a_n| > M$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

4. Two regular pentagons $ABCDE$ and $AEKPL$ are placed in space so that $\angle DAK = 60^\circ$. Prove that the planes ACK and BAL are perpendicular.

Second Day

5. *Problem 5 for Grade 9.*

6. Solve in the real numbers the system

$$\begin{aligned} x_1 + x_2 + \cdots + x_{1997} &= 1997, \\ x_1^4 + x_2^4 + \cdots + x_{1997}^4 &= x_1^3 + x_2^3 + \cdots + x_{1997}^3. \end{aligned}$$

7. In a parallelogram $ABCD$, M is the midpoint of BC and N an arbitrary point on the side AD. Let P be the intersection of MN and AC, and Q the intersection of AM and BN. Prove that the triangles BDQ and DMP have equal areas.

8. Let $d(n)$ denote the largest odd divisor of a positive integer n. The function $f : \mathbb{N} \to \mathbb{N}$ is defined by $f(2n-1) = 2^n$ and $f(2n) = n + \dfrac{2n}{d(n)}$ for all $n \in \mathbb{N}$. Find all natural numbers k such that

$$\underbrace{f(f(\cdots f(1)\cdots))}_{k} = 1997.$$

Grade 11

First Day

1. Let $ABCD$ be a parallelogram with $AB = 1$. Suppose that K is a point on the side AD such that $KD = 1$, $\angle ABK = 90^\circ$ and $\angle DBK = 30^\circ$. Determine AD.

2. Prove that among any four distinct numbers from the interval $(0, \pi/2)$ there are two, say x, y, such that

$$8\cos x \cos y \cos(x-y) + 1 > 4(\cos^2 x + \cos^2 y).$$

3. In space are given 1997 points. Let M and m be the largest and smallest distance between two of the given points. Prove that $M/m > 9$.

4. The equation $ax^3 + bx^2 + cx + d = 0$ has three distinct solutions. How many distinct solutions does the following equation have:

$$4(ax^3 + bx^2 + cx + d)(3ax + b) = (3ax^2 + 2bx + c)^2?$$

Second Day

5. Find all real solutions to the equation

$$\left(2+\sqrt{3}\right)^{x}+1=\left(2\sqrt{2+\sqrt{3}}\right)^{x}.$$

6. Find all functions $f:\mathbb{Q}^{+}\to\mathbb{Q}^{+}$ which satisfy the conditions:

   (i) $f(x+1)=f(x)+1$ for all $x\in\mathbb{Q}^{+}$;

   (ii) $f(x^2)=f(x)^2$ for all $x\in\mathbb{Q}^{+}$.

7. Determine the smallest natural number n such that among any n integers one can choose 18 integers whose sum is divisible by 18.

8. On the edges AB,BC,CD,DA of a parallelepiped $ABCDA_1B_1C_1D_1$ points K,L,M,N are selected, respectively. Prove that the circumcenters of the tetrahedra $A_1AKN,B_1BKL,C_1CLM,D_1DMN$ are vertices of a parallelogram.