3-rd Taiwanese Mathematical Olympiad 1994

Time: 4.5 hours each day.

First Day – Taipei, April 14, 1994

1. Let $ABCD$ be a quadrilateral with $AD = BC$ and $\angle A + \angle B = 120^\circ$. Let us draw equilateral triangles ACP, DCQ, DBR away from AB. Prove that the points P, Q, R are collinear.

2. Let a, b, c be positive real numbers and α be any real number. Denote
$$f(\alpha) = abc(a^\alpha + b^\alpha + c^\alpha),$$
$$g(\alpha) = a^{\alpha+2}(b+c-a) + b^{\alpha+2}(c+a-b) + c^{\alpha+2}(a+b-c).$$
Determine $|f(\alpha) - g(\alpha)|$.

3. Suppose a is a positive integer such that $5^{1994} - 1 \mid a$. Prove that the expression of a in the base 5 contains at least 1994 nonzero digits.

Second Day – Taipei, April 15, 1994

4. Prove that there are infinitely many positive integers n with the following property: For any n integers $a_1, a_2, \dots, a_n$ which form in arithmetic progression, both the mean and the standard deviation of the set $\{a_1, \dots, a_n\}$ are integers.

   Remark. The mean and standard deviation of the set $\{x_1, \dots, x_n\}$ are defined by $\overline{x} = \dfrac{a_1 + \cdots + a_n}{n}$ and $\sqrt{\dfrac{\sum (x_i - \overline{x})^2}{n}}$, respectively.

5. Given $X = \{0, a, b, c\}$, let $M(X) = \{f \mid f : X \to X\}$ denote the set of all functions from X into itself. An addition table on X is given as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

   (a) If $S = \{f \in M(X) \mid f(x+y+x) = f(x) + f(y) + f(x) \text{ for all } x, y \in X\}$, find the number of elements of S.

   (b) If $I = \{f \in M(X) \mid f(x+x) = f(x) + f(x) \text{ for all } x \in X\}$, find the number of elements of I.

6. For $-1 \le x \le 1$ and $n \in \mathbb{N}$, define
$$T_n(x) = \frac{1}{2^n}\left[\left(x + \sqrt{1-x^2}\right)^n + \left(x - \sqrt{1-x^2}\right)^n\right].$$

(a) Prove that $T_n(x)$ is a monic polynomial of degree n in x and that the maximum value of $|T_n(x)|$ is $1/2^{n-1}$.

(b) Suppose that $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a monic polynomial of degree n with real coefficients such that $p(x) > -1/2^{n-1}$ for all x, $-1 \leq x \leq 1$. Prove that there exists x_0, $-1 \leq x_0 \leq 1$ such that $p(x_0) \geq 1/2^{n-1}$.

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