10-th Taiwanese Mathematical Olympiad 2001

Time: 4.5 hours each day.

Part 1 – Taipei, April 1

1. Let $A = \{a_1, a_2, \dots, a_n\}$ be a set of $n \geq 3$ distinct integers, and let m and M be the minimum and maximum element of A, respectively. Suppose that there exists a polynomial $p(x)$ with integer coefficients such that

$$\begin{array}{ll} m < p(a) < M & \text{for all } a \in A, \text{ and} \\ p(m) < p(a) & \text{for all } a \in A \setminus \{m, M\}. \end{array}$$

   Prove that $n \leq 5$ and that there exist two integers b and c such that each element of A is a solution of the equation $p(x) + x^2 + bx + c = 0$.

2. Let $a_1, a_2, \dots, a_{15}$ be positive integers for which the number $a_k^{k+1} - a_k$ is not divisible by 17 for any $k = 1, \dots, 15$. Show that there are integers $b_1, b_2, \dots, b_{15}$ such that

   (i) $b_m - b_n$ is not divisible by 17 for $1 \leq m < n \leq 15$, and

   (ii) each b_i is a product of one or more terms of (a_i).

3. Let $A_1, A_2, \dots, A_n$ be distinct subsets of $\{1, 2, \dots, n\}$. Prove that there is an element x of S such that the subsets $A_1 \setminus \{x\}, \dots, A_n \setminus \{x\}$ are also distinct.

Part 2 – Taipei, April 25

4. Let Γ be the circumcircle of a fixed triangle ABC, and let M and N be the midpoints of the arcs BC and CA, respectively. For any point X on the arc AB, let O_1 and O_2 be the incenters of $\triangle XAC$ and $\triangle XBC$, and let the circumcircle of $\triangle XO_1O_2$ intersect Γ at X and Q. Prove that triangles QNO_1 and QMO_2 are similar, and find all possible locations of point Q.

5. Let x and y be distinct real numbers, and let $f(n)$ be defined by

$$f(n) = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}, \quad n \in \mathbb{N}.$$

   Prove that if $f(m), f(m+1), f(m+2), f(m+3)$ are integers for some $m \in \mathbb{N}$, then $f(n)$ is an integer for each n.

6. Suppose that $n-1$ items $A_1, A_2, \dots, A_{n-1}$ have already been arranged in the increasing order, and that another item A_n is to be inserted to preserve the order. What is the expected number of comparisons necessary to insert A_n?

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