Turkish IMO Team Selection Test 1998

First Day – April 18, 1998

1. Squares $BAXX'$ and $CAYY'$ are drawn in the exterior of a triangle ABC with $AB = AC$. Let D be the midpoint of BC, and E and F be the feet of the perpendiculars from an arbitrary point K on the segment BC to BY and CX, respectively.

   (a) Prove that $DE = DF$.

   (b) Find the locus of the midpoint of EF.

2. Let the sequence (a_n) be defined by $a_1 = t$ and $a_{n+1} = 4a_n(1 - a_n)$ for $n \geq 1$. How many possible values of t are there, if $a_{1998} = 0$?

3. Let $A = \{1, 2, 3, 4, 5\}$. Find the number of functions f from the nonempty subsets of A to A, such that $f(B) \in B$ for any $B \subset A$, and $f(B \cup C)$ is either $f(B)$ or $f(C)$ for any $B, C \subset A$.

Second Day – April 19, 1998

4. Suppose n houses are to be assigned to n people. Each person ranks the houses in the order of preference, with no ties. After the assignment is made, it is observed that every other assignment would assign to at least one person a less preferred house. Prove that there is at least one person who received the house he/she preferred most under this assignment.

5. In a triangle ABC, the circle through C touching AB at A and the circle through B touching AC at A have different radii and meet again at D. Let E be the point on the ray AB such that $AB = BE$. The circle through A, D, E intersect the ray CA again at F. Prove that $AF = AC$.

6. Let $f(x_1, x_2, \dots, x_n)$ be a polynomial with integer coefficients of degree less than n. Prove that if N is the number of n-tuples $(x_1, \dots, x_n)$ with $0 \leq x_i < 13$ and $f(x_1, \dots, x_n) = 0$ (mod 13), then N is divisible by 13.

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