Turkish IMO Team Selection Test 2009

First Day

1. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Z}$ such that $f(1/x) = f(x)$ and $(x+1)f(x-1) = xf(x)$ for all rational numbers $x > 1$.

2. Let $ABCD$ be a quadrilateral inscribed in a circle $k(O,r)$. Assume that AB intersects CD at P; AD intersects BC at Q; and assume that the diagonals AC and BD intersect at K. If the distance from O to the line PQ is k, prove that $OK \cdot k = r^2$.

3. In a group of 2009 people, each two have exactly one common friend. Find the least value of the difference of friends between the person with the maximal number of friends and the person with the minimal number of friends.

Second Day

4. Find all prime numbers p for which there exists a polynomial Q with integral coefficients such that the equation

$$1 + p + Q(x) \cdot Q(x^2) \cdots Q(x^{2p-2}) = 0$$

has integer solution.

5. The incircle of $\triangle ABC$ touches the sides AB, AC, and BC at C_1, B_1, and A_1 respectively. Prove that

$$\sqrt{\frac{AB_1}{AB}} + \sqrt{\frac{BC_1}{BC}} + \sqrt{\frac{CA_1}{CA}} \le \frac{3}{\sqrt{2}}.$$

6. There are $n \ge 4$ students in a class. Some of the students are friends and friendship is symmetric relation. $n-1$ students can be seated in a round table such that every student is sitting next to a his/her friends (on both sides), but n students can't be seated in that way. Prove that $n \ge 10$.

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