6-th Turkish Mathematical Olympiad 1998/99

Second Round

First Day – December 11, 1998

1. Let D be the point on the base BC of an isosceles triangle ABC such that $BD : DC = 2 : 1$, and let P be the point on the segment AD such that $\angle BAC = \angle BPD$. Prove that $\angle DPC = \frac{1}{2}\angle BAC$.

2. If $0 \le a \le b \le c$ are real numbers, prove that
$$(a+3b)(b+4c)(c+2a) \ge 60abc.$$

3. The points of a circle are colored by three colors. Prove that there exist infinitely many isosceles triangles inscribed in the circle whose vertices are of the same color.

Second Day – December 12, 1998

4. Find all positive integers x and n such that $x^3 + 3367 = 2^n$.

5. Variable points M and N are considered on the arms OX and OY, respectively, of an angle XOY so that $OM + ON$ is constant. Determine the locus of the midpoint of MN.

6. Some of the vertices of unit squares of an $n \times n$ chessboard are colored so that any $k \times k$ square consisting of these unit squares has a colored point on at least one of its sides. Let $l(n)$ denote the minimum number of colored points required to satisfy this condition. Prove that
$$\lim_{n\to\infty} \frac{l(n)}{n^2} = \frac{2}{7}.$$

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