5-th Turkish Mathematical Olympiad 1997/98

Second Round

First Day – December 12, 1997

1. Find all pairs of integers (x,y) such that $5x^2-6xy+7y^2=383$.

2. Let F be a point inside a convex pentagon $ABCDE$, and let a_1,a_2,a_3,a_4, a_5 denote the distances from F to the lines AB,BC,CD,DE,EA, respectively. The points F_1,F_2,F_3,F_4,F_5 are chosen on the inner bisectors of the angles A,B,C,D,E of the pentagon respectively, so that $AF_1=AF$, $BF_2=BF$, $CF_3=CF$, $DF_4=DF$ and $EF_5=EF$. If the distances from F_1,F_2,F_3,F_4,F_5 to the lines EA,AB,BC,CD,DE are b_1,b_2,b_3,b_4,b_5, respectively, prove that
$$a_1+a_2+a_3+a_4+a_5 \le b_1+b_2+b_3+b_4+b_5.$$

3. Let n and k be positive integers, where $n>1$ is odd. Suppose n voters are to elect one of the k cadidates from a set A according to the rule of *majoritarian compromise* described below.

   After each voter ranks the candidates in a column according to his/her preferences, these columns are concatenated to form a $k\times n$ voting matrix. We denote the number of occurences of $a\in A$ in the i-th row of the voting matrix by a_i. Let l_a stand for the minimum integer l for which $\sum_{i=1}^{l} a_i > n/2$. Setting $l' = \min\{l_a \mid a\in A\}$, we will regard the voting matrices which make the set $\{a\in A \mid l_a = l'\}$ as admissible. For each such matrix, the single candidate in this set will get elected according to majoritarian compromise. Moreover, if $\omega_1 \ge \omega_2 \ge \cdot \ge \omega_k \ge 0$ are given, for each admissible voting matrix, $\sum_{i=1}^{k}\omega_i a_i$ is called the total weighted score of $a\in A$. We will say that the system $(\omega_1,\dots,\omega_k)$ of weights represents majoritarian compromise if the total score of the elected candidate is maximum among the scores of all candidates.

   (a) Determine whether there is a system of weights representing majoritarian compromise if $k=3$.

   (b) Show that such a system of weights does not exist for $k>3$.

Second Day – December 13, 1997

4. Let $e>0$ be a given real number. Find the least value of $f(e)$ (in terms of e only) such that the inequality
$$a^3+b^3+c^3+d^3 \le e^2(a^2+b^2+c^2+d^2)+f(e)(a^4+b^4+c^4+d^4)$$
holds for all real numbers a,b,c,d.

5. In a triangle ABC, the inner and outer bisectors of the angle A meet the line BC at D and E, respectively. Let d be a common tangent of the circumcircle of $\triangle ABC$ and the circle with diameter DE and center F. The projections of the tangency points onto FO are denoted by P and Q, and the length of the common tangent is denoted by m. Prove that $PQ = m$. (Wrong!)

6. Let $D_1, D_2, \dots, D_n$ be rectangular parallelepipeds in space, with edges parallel to the x,y,z axes. For each D_i, let x_i, y_i, z_i be the lengths of its projections onto the x,y,z axes, respectively. Suppose that for all pairs D_i, D_j, if at least one of $x_i < x_j$, $y_i < y_j$, $z_i < z_j$ holds, then $x_i \le x_j$, $y_i \le y_j$, and $z_i < z_j$. If the volume of the region $\bigcup_{i=1}^{n} D_i$ equals 1997, prove that there is a subset $\{D_{i_1}, D_{i_2}, \dots, D_{i_m}\}$ of the set $\{D_1, \dots, D_n\}$ such that

   (i) if $k \ne l$ then $D_{i_k} \cap D_{i_l} = \emptyset$, and

   (ii) the volume of $\bigcup_{k=1}^{m} D_{i_k}$ is at least 73.

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