3-rd Turkish Mathematical Olympiad 1995/96

Second Round

First Day – December 8, 1995

1. Let $m_1, m_2, \ldots, m_k$ be integers with $2 \le m_1$ and $2m_i \le m_{i+1}$ for all i. Show that for any integers $a_1, a_2, \ldots, a_k$ there are infinitely many integers x which do not satisfy any of the congruences
$$x \equiv a_i \pmod{m_i}, \quad i = 1, 2, \ldots, k.$$

2. Let ABC be an acute triangle and let k_1, k_2, k_3 be the circles with diameters BC, CA, AB, respectively. Let K be the radical center of these circles. Segments AK, BK, CK meet k_1, k_2, k_3 again at D, E, F, respectively. If the areas of triangles ABC, DBC, ECA, FAB are u, x, y, z respectively, prove that
$$u^2 = x^2 + y^2 + z^2.$$

3. Let A be a real number and (a_n) be a sequence of real numbers such that $a_1 = 1$ and
$$1 < \frac{a_{n+1}}{a_n} \le A \quad \text{for all } n \in \mathbb{N}.$$
   (a) Show that there is a unique non-decreasing surjective function $f : \mathbb{N} \to \mathbb{N}$ such that $1 < A^{k(n)}/a_n \le A$ for all $n \in \mathbb{N}$.

   (b) If k takes every value at most m times, show that there is a real number $C > 1$ such that $Aa_n \ge C^n$ for all $n \in \mathbb{N}$.

Second Day – December 9, 1995

4. In a triangle ABC with $AB \ne AC$, the internal and external bisectors of the angle A meet the line BC at D and E respectively. If the feet of the perpendiculars from a point F on the circle with diameter DE to BC, CA, AB are K, L, M, respectively, show that $KL = KM$.

5. Let $t(A)$ denote the sum of elements of a nonempty set A of integers, and define $t(\emptyset) = 0$. Find a set X of positive integers such that for every integer k there is a unique ordered pair of disjoint subsets (A_k, B_k) of X such that $t(A_k) - t(B_k) = k$.

6. Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$,
$$f(m) \mid f(n) \quad \text{if and only if} \quad m \mid n.$$

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