15-th Turkish Mathematical Olympiad 2007/08

First Day

1. Given an acute-angled triangle ABC, denote by O and H its circumcenter and orthocenter, respectively. Denote by A_1, B_1, C_1 the midpoints of the sides BC, CA, and AB, respectively. The rays HA_1, HB_1, and HC_1 respectively intersect the circumcircle of $\triangle ABC$ at A_0, B_0, and C_0. Prove that O, H, and H_0 are collinear if and only if H_0 is the orthocenter of $A_0B_0C_0$.

2. (a) Find all primes p such that $\frac{7^{p-1}-1}{p}$ is a perfect square.

   (b) FInd all primes p such that $\frac{11^{p-1}-1}{p}$ is a perfect square.

3. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a^2b^2}{c^3(a^2-ab+b^2)}+\frac{b^2c^2}{a^3(b^2-bc+c^2)}+\frac{c^2a^2}{b^3(a^2-ca+c^2)} \geq \frac{3}{ab+bc+ca}.$$

Second Day

4. Assume that the function $f : \mathbb{N}\times\mathbb{Z}\to\mathbb{Z}$ satisfies the following conditions:

   (i) $f(0,0)=1$, $f(0,1)=1$,

   (ii) For all $k\in\mathbb{Z}\setminus\{0,1\}$: $f(0,k)=0$, and

   (iii) For all $n\geq 1$ and all $k\in\mathbb{Z}$: $f(n,k)=f(n-1,k)+f(n-1,k-2n)$.

   Evaluate the sum:

$$\sum_{k=0}^{\binom{2009}{2}} f(2008,k).$$

5. Assume that a line l lies in the plane of a circle Γ and does not have common points with Γ. Determine the locus of points of the intersection of circles with diameter AB for all $A,B\in l$ for which there exist P, Q, R, $S\in\Gamma$ such that $PQ\cap RS=A$, $PS\cap QR=B$.

6. A computer network with 2008 computers has a form of a graph in which any of the two cycles dont share any common vertex. A hacker and an administrator are playing the following game: On the first move hacker selects one computer and hacks it, on the second move administrator selects another computer and protects it. On every $2k+1$-th move hacker hacks one more computer (if he can) which wasnt protected by the administrator and is directly connected (with an edge) to a computer which was hacked by the hacker before. On every $2k+2$-th move administrator protects one more computer (if he can) which wasnt hacked by the hacker and is directly connected (with an edge) to a computer which was

protected by the administrator before for every $k > 0$. If both of them cant make move, the game ends. Determine the maximum number of computers which the hacker can guarantee to hack at the end of the game.

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