# 10-th Turkish Mathematical Olympiad 2002/03

## Second Round

*First Day – December 14, 2002*

1. Let $(a_1, a_2, \dots, a_n)$ be a permutation of $1, 2, \dots, n$, where $n \geq 2$. For each $k = 1, \dots, n$, $a_k$ apples are placed at the point $k$ on the real axis. Children named $A, B, C$ are assigned respective points $x_A, x_B, x_C \in \{1, \dots, n\}$. For each $k$, the children whose points are closest to $k$ divide $a_k$ apples equally among themselves. We call $(x_A, x_B, x_C)$ a *stable configuration* if no child's total share can be increased by assigning a new point to this child and not changing the points of the other two. Determine the values of $n$ for which a stable configuration exists for some distribution $(a_1, \dots, a_n)$ of the apples.

2. Two circles are externally tangent to each other at a point $A$ and internally tangent to a third circle $\Gamma$ at points $B$ and $C$. Let $D$ be the midpoint of the secant of $\Gamma$ which is tangent to the smaller circles at $A$. Show that $A$ is the incenter of the triangle $BCD$ if the centers of the circles are not collinear.

3. Graph Airlines (GA) operates flights between some of the cities of the Republic of Graphia. There are GA flights between each city and at least three different cities, and it is possible to travel from any city in Graphia using GA flights. GA decides to discontinue some of its flights. Show that this can be done in such a way that it is still possible to travel between any two cities using GA flights, yet at least $2/9$ of the cities have only one flight.

*Second Day – December 15, 2002*

4. Find all prime numbers $p$ for which the number of ordered pairs of integers $(x, y)$ with $0 \leq x, y < p$ satisfying the condition $y^2 \equiv x^3 - x \pmod{p}$ is exactly $p$.

5. In an acute triangle $ABC$ with $BC < AC < AB$, the points $D$ on side $AB$ and $E$ on side $AC$ satisfy the condition $DB = BC = CE$. Show that the circumradius of the triangle $ADE$ is equal to the distance between the incenter and the circumcenter of the triangle $ABC$.

6. Let $n$ be a positive integer and let $T$ denote the collection of points $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ for which there exists a permutation $\sigma$ of $1, 2, \dots, n$ such that $x_{\sigma(i)} - x_{\sigma(i+1)} \geq 1$ for each $i = 1, \dots, n-1$. Prove that there is a real number $d$ satisfying the following condition: For every $(a_1, \dots, a_n) \in \mathbb{R}^n$ there exist points $(b_1, \dots, b_n)$ and $(c_1, \dots, c_n)$ in $T$ such that, for each $i = 1, \dots, n$,

$$a_i = \frac{1}{2}(b_i + c_i), \quad |a_i - b_i| \leq d, \quad \text{and} \quad |a_i - c_i| \leq d.$$



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