Swiss IMO Team Selection Tests 1999

First Test

May 17

1. Two circles intersect at points M and N. Let A be a point on the first circle, distinct from M,N. The lines AM and AN meet the second circle again at B and C, respectively. Prove that the tangent to the first circle at A is parallel to BC.

2. Can the set $\{1,2,\dots,33\}$ be partitioned into 11 three-element sets, in each of which one element equals the sum of the other two?

3. Find all functions $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ that satisfy
$$\frac{1}{x}f(-x)+f\left(\frac{1}{x}\right)=x \quad \text{for all } x\neq 0.$$

4. Find all real solutions (x,y,z) of the system
$$\frac{4x^2}{1+4x^2}=y, \quad \frac{4y^2}{1+4y^2}=z, \quad \frac{4z^2}{1+4z^2}=x.$$

5. In a rectangle $ABCD$, M and N are the midpoints of AD and BC respectively and P is a point on line CD. The line PM meets AC at Q. Prove that MN bisects the angle $\angle QNP$.

Second Test

May 20

1. Prove that if m and n are positive integers such that m^2+n^2-m is divisible by $2mn$, then m is a perfect square.

2. A square is dissected into rectangles with sides parallel to the sides of the square. For each of these rectangles, the ratio of its shorter side to its longer side is considered. Show that the sum of all these ratios is at least 1.

3. Find all n for which there are real numbers $0<a_1\le a_2\le\cdots\le a_n$ with
$$\sum_{k=1}^{n}a_k=96, \quad \sum_{k=1}^{n}a_k^2=144, \quad \sum_{k=1}^{n}a_k^3=216.$$

4. Prove that for every polynomial $P(x)$ of degree 10 with integer coefficients there is an infinite arithmetic progression (in both directions) which does not contain any number $P(k)$, $k\in\mathbb{Z}$.

5. Prove that the product of five consecutive positive integers cannot be a perfect square.

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