Swiss IMO Team Selection Tests 2002

First Test

May 10

1. In space are given 24 points, no three of which are collinear. Suppose that there are exactly 2002 planes determined by three of these points. Prove that there is a plane containing at least six points.

2. A point O inside a parallelogram $ABCD$ satisfies $\angle AOB + \angle COD = \pi$. Prove that $\angle CBO = \angle CDO$.

3. Let $1 = d_1, d_2, d_3, d_4$ be the four smallest divisors of a positive integer n (having at least four divisors). Find all n such that $d_1^2 + d_2^2 + d_3^2 + d_4^2 = n$.

4. A 7×7 square is divided into unit squares by lines parallel to its sides. Some Swiss crosses (obtained by removing corner unit squares from a square of side 3) are to be put on the large square, with the edges along division lines. Find the smallest number of unit squares that need to be marked in such a way that every cross covers at least one marked square.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

   (a) The set $\{ f(x)/x \mid x \in \mathbb{R}, x \neq 0\}$ is finite, and

   (b) $f(x - 1 - f(x)) = f(x) - 1 - x$ for all real x.

Second Test

May 25

1. A sequence $x_1, x_2, x_3, \dots$ has the following properties:

   (a) $1 = x_1 < x_2 < x_3 < \cdots$;

   (b) $x_{n+1} \leq 2n$ for all $n \in \mathbb{N}$.

   Prove that for each positive integer k there exist indices i and j such that $k = x_i - x_j$.

2. Let P be an interior point of an equilateral triangle ABC and let X, Y, Z be the projections of P on the sides BC, CA, AB, respectively. Prove that the sum of the areas of triangles BXP, CYP, and AZP does not depend on P.

3. In a group of n persons, every weekend someone organizes a party for which he invites all his acquaintances. Those who meet at a party become acquainted. After each of the n persons has organized a party, there still are two persons not knowing each other. Show that these two persons will never get acquainted at such a party. (Acquaintance is a symmetric relation.)

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4. For each real number a and integer $n \geq 1$ prove the inequality

$$a^n + \frac{1}{a^n} - 2 \geq n^2 \left(a + \frac{1}{a} - 2 \right)$$

and find the cases of equality.

5. Given an integer $m \geq 2$, find the smallest integer $k > m$ such that for any partition of the set $\{m, m+1, \dots, k\}$ into two classes A and B at least one of the classes contains three numbers a, b, c (not necessarily distinct) such that $a^b = c$.