Swiss IMO Team Selection Tests 2000

First Test

April 28

1. A convex quadrilateral $ABCD$ is inscribed in a circle. Show that the line connecting the midpoints of the arcs AB and CD and the line connecting the midpoints of the arcs BC and DA are perpendicular.

2. Real numbers $a_1, a_2, \dots, a_{16}$ satisfy the conditions
$$\sum_{i=1}^{16} a_i = 100 \quad \text{and} \quad \sum_{i=1}^{16} a_i^2 = 1000.$$
What is the greatest possible value of a_{16}?

3. An equilateral triangle of side 1 is covered by five congruent equilateral triangles of side $s < 1$ with sides parallel to those of the larger triangle. Show that some four of these smaller triangles also cover the large triangle.

4. Let $q(n)$ denote the sum of the digits of a natural number n. Determine $q(q(q(2000^{2000})))$.

5. Consider all words of length n consisting of the letters I, O, M. How many such words are there, which contain no two consecutive M's?

Re-Test

April 28

1. Positive real numbers x, y, z have the sum 1. Prove that
$$\sqrt{7x+3} + \sqrt{7y+3} + \sqrt{7z+3} \le 7.$$
Can number 7 on the right hand side be replaced with a smaller constant?

2. Show that the equation $14x^2 + 15y^2 = 7^{2000}$ has no integer solutions.

3. Let $f(x) = \frac{4^x}{4^x+2}$ for $x > 0$. Evaluate
$$\sum_{k=1}^{1290} f\left(\frac{k}{1291}\right).$$

4. Two given circles k_1 and k_2 intersect at points P and Q. Construct a segment AB through P with the endpoints at k_1 and k_2 for which $AP \cdot PB$ is maximal.

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5. At n distinct points of a circular race course there are n cars ready to start. Each car moves at a constant speed and covers the circle in an hour. On hearing the initial signal, each of them selects a direction and starts moving immediately. If two cars meet, both of them change directions and go on without loss of speed. Show that at a certain moment each car will be at its starting point.

Second Test

May 20

1. The vertices of a regular $2n$-gon ($n \geq 3$) are labelled with the numbers $1, 2, \ldots, 2n$ so that the sum of the numbers at any two adjacent vertices equals the sum of the numbers at the vertices diametrically opposite to them. Show that this is only possible if n is odd.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, y,

$$f(f(x)+y) = f(x^2 - y) + 4yf(x).$$

3. The incircle of a triangle ABC touches the sides AB, BC, CA at points D, E, F respectively. Let P be an internal point of triangle ABC such that the incircle of triangle ABP touches AB at D and the sides AP and BP at Q and R. Prove that the points E, F, R, Q lie on a circle.

4. The polynomial P of degree n satisfies $P(k) = \dfrac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$. Find $P(n+1)$.

5. Let $S = \{P_1, P_2, \ldots, P_{2000}\}$ be a set of 2000 points in the interior of a circle of radius 1, one of which at its center. For $i = 1, 2, \ldots, 2000$ denote by x_i the distance from P_i to the closest point $P_j \neq P_i$. Prove that

$$x_1^2 + x_2^2 + \cdots + x_{2000}^2.$$