26-th Swedish Mathematical Competition 1986

Final Round

November 22, 1986

1. Show that the polynomial $x^6 - x^5 + x^4 - x^3 + x^2 - x + \frac{3}{4}$ has no real zeroes.

2. The diagonals AC and BD of a quadrilateral $ABCD$ intersect at O. If S_1 and S_2 are the areas of triangles AOB and COD and S that of $ABCD$, show that
$$\sqrt{S_1} + \sqrt{S_2} \leq \sqrt{S}.$$
Prove that equality holds if and only if AB and CD are parallel.

3. Let $N \geq 3$ be a positive integer. For every pair (a, b) of integers with $1 \leq a < b \leq N$ consider the quotient $q = b/a$. Show that the pairs with $q < 2$ are equally numbered as those with $q > 2$.

4. Prove that $x = y = z = 1$ is the only positive solution of the system
$$\begin{aligned} x + y^2 + z^3 &= 3 \\ y + z^2 + x^3 &= 3 \\ z + x^2 + y^3 &= 3. \end{aligned}$$

5. In the arrangement of pn real numbers below, the difference between the greatest and smallest numbers in each row is at most d, $d > 0$.
$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}$$
Prove that, when the numbers in each column are rearranged in decreasing order, the difference between the greatest and smallest numbers in each row will still be at most d.

6. The interval $[0, 1]$ is covered by a finite number of intervals. Show that one can choose a number of these intervals which are pairwise disjoint and have the total length at least $1/2$.

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